

Soustavy obvyklých lineárních diferenciálních

rovníc

Příklad 1

$$a) \quad \vec{y}' = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \vec{y}$$

$$\begin{vmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)(-1-\lambda) + 3 \cdot 4 + 2 \cdot 1 - \\ - 4 \cdot 2 \cdot (2-\lambda) + (1-\lambda) + 3(-1-\lambda) = \\ = -(1-\lambda)(2-\lambda)(1+\lambda) + 12 + 2 - 8(2-\lambda) + 1 - \lambda - 3 - 3\lambda = \\ = -(1-\lambda)(2-\lambda)(1+\lambda) - 4 + 4\lambda = -(1-\lambda)(2-\lambda)(1+\lambda) - 4(1-\lambda) = \\ = (1-\lambda)[-(2-\lambda)(1+\lambda) - 4] = (1-\lambda)(-\lambda^2 - \lambda + 2 - 4) = \\ = (1-\lambda)(-\lambda^2 - \lambda - 2) = (1-\lambda)(\lambda^2 + \lambda + 2)$$

$$\lambda_1 = 1, \lambda_2 = -3, \lambda_3 = -2$$

$$\lambda_1 = 1: \begin{pmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{pmatrix} \vec{v}_1 = \vec{0}$$

$$\begin{pmatrix} 3 & 1 & -1 \\ 0 & -1 & 4 \\ 2 & 1 & -2 \end{pmatrix} \xrightarrow{(3) \cdot 2(1)} \begin{pmatrix} 3 & 1 & -1 \\ 0 & -1 & 4 \\ 0 & 1 & -4 \end{pmatrix} \sim \begin{pmatrix} 3 & 1 & -1 \\ 0 & -1 & 4 \\ 0 & -1 & 4 \end{pmatrix}$$

$$\begin{cases} 3x + y - z = 0 \\ -y + 4z = 0 \end{cases} \Rightarrow \begin{cases} z = t, t \in \mathbb{R} - \{0\} \\ y = 4z = 4t \\ x = \frac{z - y}{3} = -t \end{cases} \Rightarrow \vec{v}_1 = t \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}, t=1: \\ \vec{v}_1 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -3: \begin{pmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{pmatrix} \vec{v}_2 = \vec{0}$$

$$\begin{pmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{pmatrix} \xrightarrow{2(2) + 3(1)} \begin{pmatrix} -2 & -1 & 4 \\ 0 & -5 & 10 \\ 2 & 1 & -4 \end{pmatrix} \sim \begin{pmatrix} -2 & -1 & 4 \\ 0 & 1 & -2 \\ 2 & 1 & -4 \end{pmatrix}$$

(1)

$$\left. \begin{aligned} -2x - y + 4z &= 0 \\ y - 2z &= 0 \end{aligned} \Rightarrow \begin{aligned} z &= t, t \in \mathbb{R} - \{0\} \\ y &= 2z = 2t \\ x &= \frac{4z - y}{2} = t \end{aligned} \right\} \Rightarrow \vec{v}_2 = t \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, t=1: \\ \vec{v}_2 &= \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\lambda_3 = -2:$$

$$\begin{pmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \\ 2 & 1 & 1 \end{pmatrix} \cdot \vec{v}_3 = \vec{0}$$

$$\begin{pmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \\ 2 & 1 & 1 \end{pmatrix} \xrightarrow{\substack{(2)-(1) \\ 3(3)-2(1)}}} \begin{pmatrix} 3 & -1 & 4 \\ 0 & 5 & -5 \\ 0 & 5 & -5 \end{pmatrix} \sim \begin{pmatrix} 3 & -1 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$3x - y + 4z = 0$$

$$\left. \begin{aligned} y - z &= 0 \Rightarrow z = t, t \in \mathbb{R} - \{0\} \\ y &= z = t \\ x &= \frac{y - 4z}{3} = -t \end{aligned} \right\} \Rightarrow \vec{v}_3 = t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, t=1: \\ \vec{v}_3 &= \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\vec{y}_1 = e^x \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}, \vec{y}_2 = e^{3x} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \vec{y}_3 = e^{-2x} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\vec{y} = c_1 e^x \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} + c_2 e^{3x} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + c_3 e^{-2x} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, c_1, c_2, c_3 \in \mathbb{R}$$

$$b) \vec{y}' = \begin{pmatrix} -1 & 2 & -3 \\ 5 & -3 & 2 \\ 1 & 0 & -3 \end{pmatrix} \vec{y}, \vec{y}(0) = \begin{pmatrix} 4 \\ 14 \\ 3 \end{pmatrix}$$

$$\begin{aligned} \begin{vmatrix} -1-\lambda & 2 & -3 \\ 5 & -3-\lambda & 2 \\ 1 & 0 & -3-\lambda \end{vmatrix} &= (-1-\lambda)(-3-\lambda)^2 + 4 + 3(-3-\lambda) - 2 \cdot 5 \cdot (-3-\lambda) = \\ &= -(1+\lambda)(3+\lambda)^2 + 4 - 9 - 3\lambda + 30 + 10\lambda = \\ &= -(1+\lambda)(9+6\lambda+\lambda^2) + 25 + 7\lambda = -(9+6\lambda+\lambda^2+9\lambda+6\lambda^2+\lambda^3) + \\ &+ 25 + 7\lambda = -\lambda^3 - 7\lambda^2 - 8\lambda + 16 \end{aligned}$$

(2)

$$\lambda = 1 : -1 - 7 - 8 + 16 = 0 \quad \rightarrow \dots$$

$$(-\lambda^3 - 7\lambda^2 - 8\lambda + 16) : (\lambda - 1) = -\lambda^2 - 8\lambda - 16 = -(\lambda^2 + 8\lambda + 16) = -(\lambda + 4)$$

$$\begin{array}{r} +\lambda^3 - \lambda^2 \\ \hline -8\lambda^2 - 8\lambda + 16 \\ +8\lambda^2 - 8\lambda \\ \hline -16\lambda + 16 \\ +16\lambda - 16 \\ \hline 0 \end{array}$$

$$\lambda_1 = 1, \lambda_{2,3} = -4$$

$$\lambda_1 = 1 : \begin{pmatrix} -2 & 2 & -3 \\ 5 & -4 & 2 \\ 1 & 0 & -4 \end{pmatrix} \vec{v}_1 = \vec{0}$$

$$\begin{pmatrix} 1 & 0 & -4 \\ 5 & -4 & 2 \\ -2 & 2 & -3 \end{pmatrix} \begin{array}{l} (2) - 5(1) \\ (3) + 2(1) \end{array} \sim \begin{pmatrix} 1 & 0 & -4 \\ 0 & -4 & 22 \\ 0 & 2 & -11 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -4 \\ 0 & -2 & 11 \\ 0 & 2 & -11 \end{pmatrix}$$

$$\begin{cases} x - 4z = 0 \\ -2y + 11z = 0 \end{cases} \Rightarrow \begin{cases} z = t, t \in \mathbb{R} - \{0\} \\ y = \frac{11z}{2} = \frac{11}{2}t \\ x = 4z = 4t \end{cases} \Rightarrow \begin{cases} \vec{v}_1 = t \begin{pmatrix} 4 \\ \frac{11}{2} \\ 1 \end{pmatrix}, t = 2 : \\ \vec{v}_1 = \begin{pmatrix} 8 \\ 11 \\ 2 \end{pmatrix} \end{cases}$$

$$\lambda_2 = -4 : \begin{pmatrix} 3 & 2 & -3 \\ 5 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \vec{v}_2 = \vec{0}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 5 & 1 & 2 \\ 3 & 2 & -3 \end{pmatrix} \begin{array}{l} (2) - 5(1) \\ (3) - 3(1) \end{array} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 2 & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \end{pmatrix}$$

$$\begin{cases} x + z = 0 \\ y - 3z = 0 \end{cases} \Rightarrow \begin{cases} z = t, t \in \mathbb{R} - \{0\} \\ y = 3z = 3t \\ x = -z = -t \end{cases} \Rightarrow \begin{cases} \vec{v}_2 = t \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}, t = 1 : \\ \vec{v}_2 = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \end{cases}$$

$$\lambda_3 = -4 : (A - \lambda E)^{-1} = \begin{pmatrix} 5 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 2 & -3 \\ 5 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 16 & 0 & -8 \\ 22 & 11 & -11 \\ 4 & 2 & -2 \end{pmatrix}$$

hledáme sobecněny vlastni vektor:

$$\begin{pmatrix} 16 & 8 & -8 \\ 22 & 11 & -11 \\ 4 & 2 & -2 \end{pmatrix} \vec{v}_3 = \vec{0}$$

$$\begin{pmatrix} 2 & 1 & -1 \\ 2 & 1 & -1 \\ 2 & 1 & -1 \end{pmatrix} \sim (2 \ 1 \ -1) \Rightarrow 2x + y - z = 0$$

$$\Rightarrow \left. \begin{aligned} x &= t, t \in \mathbb{R} - \{0\} \\ y &= s, s \in \mathbb{R} - \{0\} \\ z &= 2x + y = 2t + s \end{aligned} \right\} \Rightarrow$$

$$\Rightarrow \vec{v}_3 = \begin{pmatrix} t \\ s \\ 2t + s \end{pmatrix}, t=0, s=1 \Rightarrow$$

$$\Rightarrow \vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$(A - \lambda E) \vec{v}_3 \neq \vec{0} : \begin{pmatrix} 3 & 2 & -3 \\ 5 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{y}_1 = e^x \begin{pmatrix} 8 \\ 11 \\ 2 \end{pmatrix}, \vec{y}_2 = e^{-4x} \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}, \vec{y}_3 = e^{-4x} [E + x(A - \lambda E)] \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} =$$

$$= e^{-4x} \begin{pmatrix} 1+3x & 2x & -3x \\ 5x & x+1 & 2x \\ 1x & 0 & x+1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = e^{-4x} \begin{pmatrix} -x \\ 3x+1 \\ x+1 \end{pmatrix}$$

$$\vec{y} = c_1 e^x \begin{pmatrix} 8 \\ 11 \\ 2 \end{pmatrix} + c_2 e^{-4x} \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} + c_3 e^{-4x} \begin{pmatrix} -x \\ 3x+1 \\ x+1 \end{pmatrix}$$

dopouštáme c_1, c_2, c_3 z počáteční podmínky:

$$\vec{y}(0) = \begin{pmatrix} 7 \\ 14 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} 8 \\ 11 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$7 = 8c_1 - c_2 \Rightarrow c_2 = 8c_1 - 7$$

$$14 = 11c_1 + 3c_2 + c_3$$

$$3 = 2c_1 + c_2 + c_3$$

$$14 = 11c_1 + 24c_1 - 21 + c_3$$

$$3 = 2c_1 + 8c_1 - 7 + c_3$$

$$35 = 35c_1 + c_3$$

$$10 = 10c_1 + c_3 \quad (-)$$

$$25 = 25c_1 \Rightarrow c_1 = 1$$

$$c_3 = 10 - 10c_1 = 0$$

$$c_2 = 8c_1 - 7 = 1$$

$$\underline{\underline{\vec{y} = e^x \begin{pmatrix} 8 \\ 11 \\ 2 \end{pmatrix} + e^{-4x} \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}}}$$

$$e) \vec{y}' = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix} \vec{y}$$

$$\begin{vmatrix} -\lambda & 0 & -2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix} = \lambda^2(2-\lambda) + 2 \cdot 2 \cdot (2-\lambda) = \lambda^2(2-\lambda) + 4(2-\lambda) =$$

$$= (\lambda^2 + 4)(2-\lambda)$$

$$\lambda_1 = 2, \lambda_{2,3} = \pm 2i$$

$$\lambda_1 = 2: \begin{pmatrix} -2 & 0 & -2 \\ 0 & 0 & 0 \\ 2 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\left. \begin{array}{l} x+z=0 \\ -2z=0 \Rightarrow z=0 \\ x=-z=0 \\ y=t, t \in \mathbb{R} - \{0\} \end{array} \right\} \Rightarrow \vec{v}_1 = t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, t=1:$$

$$\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_2 = 2i : \begin{pmatrix} -2i & 0 & -2 \\ 0 & 2-2i & 0 \\ 2 & 0 & -2i \end{pmatrix} \vec{v}_2 = \vec{0}$$

$$\begin{pmatrix} i & 0 & 1 \\ 0 & 2-2i & 0 \\ 1 & 0 & -i \end{pmatrix} \sim \begin{pmatrix} i & 0 & 1 \\ 0 & 2-2i & 0 \\ 0 & 0 & -i^2-1 \end{pmatrix} \sim \begin{pmatrix} i & 0 & 1 \\ 0 & 2-2i & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} i & 0 & 1 \\ 0 & 2-2i & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} ix + z &= 0 \\ (2-2i) \cdot y &= 0 \Rightarrow y = 0 \\ x &= t, t \in \mathbb{C} - \{0\} \\ z &= -ix = -it \end{aligned} \Rightarrow \vec{v}_2 = t \begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix}, t=1: \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix}$$

$$\lambda_3 = -2i : \vec{v}_3 = \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$$

$$\vec{y}_1 = e^{2ix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \vec{y}_2 = e^{2ix} \begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix}, \vec{y}_3 = e^{-2ix} \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$$

Euler. vltah: $e^{\lambda x} = e^{(\alpha+i\beta)x} = e^{\alpha x} \cdot (\cos \beta x + i \sin \beta x)$

$$\begin{aligned} \frac{\vec{y}_2 + \vec{y}_3}{2} &= \frac{1}{2} \left[e^{2ix} \begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix} + e^{-2ix} \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix} \right] = \\ &= \frac{1}{2} \left[(\cos 2x + i \sin 2x) \cdot \begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix} + (\cos(-2x) + i \sin(-2x)) \cdot \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix} \right] = \\ &= \frac{1}{2} \left\{ (\cos 2x + i \sin 2x) \cdot \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right] + \right. \\ &\quad \left. + (\cos 2x - i \sin 2x) \cdot \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - i \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right] \right\} = \\ &= \frac{1}{2} \left[\cos 2x \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + i \cdot \cos 2x \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + i \cdot \sin 2x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \sin 2x \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + \right. \\ &\quad \left. + \cos 2x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - i \cos 2x \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} - i \sin 2x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \sin 2x \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right] = \\ &= \cos 2x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \sin 2x \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \text{Re}(\vec{y}_2) \end{aligned}$$

$$\frac{y_2 - y_3}{2i} = \frac{1}{2i} \left[2i \cos 2x \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + 2i \sin 2x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] =$$

$$= \cos 2x \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + \sin 2x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \operatorname{Im}(y_2)$$

$$\underline{\underline{\vec{y} = c_1 e^{2x} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \left[\cos 2x \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \sin 2x \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right] + c_3 \left[\cos 2x \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + \sin 2x \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right]}}$$

Prüklad 2

$$a) \vec{y}' = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 2 \end{pmatrix} \vec{y} + \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} e^{-x}, \quad \vec{y}(0) = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & -\lambda & 2 \\ 1 & 0 & 2-\lambda \end{vmatrix} = -\lambda(2-\lambda)^2$$

$$\lambda_1 = 0, \lambda_{2,3} = 2$$

$$\lambda_1 = 0: \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 2 \end{pmatrix} \vec{v}_1 = \vec{0}$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 2 \end{pmatrix} \xrightarrow{R_2 - R_1} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 4 \\ 1 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left. \begin{array}{l} x = 0 \\ z = 0 \\ v = t, t \in \mathbb{K} - \{0\} \end{array} \right\} \Rightarrow \vec{v}_1 = t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, t = 1: \vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_2 = 2: \begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 2 \\ 1 & 0 & 0 \end{pmatrix} \vec{v}_2 = \vec{0}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\left. \begin{array}{l} x=0 \\ y-z=0 \Rightarrow z=t, t \in \mathbb{R} - \{0\} \\ y=z=t \end{array} \right\} \Rightarrow \vec{v}_2 = t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, t=1: \\ \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$(A - \lambda E)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 2 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 2 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & -4 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & -4 \\ 0 & 0 & 0 \end{pmatrix} \vec{v}_3 = \vec{0} \Rightarrow \begin{array}{l} 4y - 4z = 0 \\ y - z = 0 \Rightarrow z = t, t \in \mathbb{R} \\ y = z = t \\ x = s, s \in \mathbb{R} \end{array} \Rightarrow \vec{v}_3 = \begin{pmatrix} s \\ t \\ t \end{pmatrix}$$

$s=1, t=0 : \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

ale $(A - \lambda E) \vec{v}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 2 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \neq \vec{0}$

$$\vec{y}_1 = e^{0x} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \vec{y}_2 = e^{2x} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \vec{y}_3 = e^{2x} [E + x(A - \lambda E)] \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} =$$

$$= e^{2x} \begin{pmatrix} 1 & 0 & 0 \\ x & 1-2x & 2x \\ x & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = e^{2x} \begin{pmatrix} 1 \\ x \\ x \end{pmatrix}$$

$$\vec{y}_H = c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{2x} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{2x} \begin{pmatrix} 1 \\ x \\ x \end{pmatrix}$$

$\lambda = -1$ je kořenem char. polynomu matice? \underline{NE}
 a složky vektoru \vec{b} jsou polynomy stupně nejvýše 0 \Rightarrow
 $\vec{y}_p = e^{-x} \cdot \begin{pmatrix} A \\ B \\ C \end{pmatrix}, \vec{y}_p = -e^{-x} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} -A \\ -B \\ -C \end{pmatrix} e^{-x}$

$$\begin{pmatrix} -A \\ -B \\ -C \end{pmatrix} e^{-x} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} A \\ B \\ C \end{pmatrix} \cdot e^{-x} + \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} e^{-x}$$

$1: e^{-x}$ to
 $\forall x \in \mathbb{R}$

$$\begin{pmatrix} -A \\ -B \\ -C \end{pmatrix} = \begin{pmatrix} 2A \\ A+2C \\ A+2C \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$-A = 2A + 3$$

$$-B = A + 2C + 2$$

$$-C = A + 2C + 1$$

$$-3A = 3 \Rightarrow \boxed{A = -1}$$

$$-A - B - 2C = 2 \Rightarrow 1 - B - 2C = 2 \Rightarrow B + 2C = -1 \Rightarrow \boxed{B = -1 - 2C = -1}$$

$$-A - 3C = 1 \Rightarrow \boxed{C = \frac{A+1}{-3} = 0}$$

$$\underline{\vec{y}_p = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} e^{-x}}$$

$$\vec{y} = c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{2x} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{2x} \begin{pmatrix} 1 \\ x \\ x \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} e^{-x}$$

$$\vec{y}(0) = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$$

$$2 = c_3 - 1 \Rightarrow \boxed{c_3 = 3}$$

$$3 = c_1 + c_2 - 1 = \boxed{c_1 = 4 - c_2 = 3}$$

$$\boxed{1 = c_2}$$

$$\underline{\vec{y} = 3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + e^{2x} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 3e^{2x} \begin{pmatrix} 1 \\ x \\ x \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} e^{-x}}$$

$$b) \vec{y}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \vec{y} + \begin{pmatrix} -x^2 \\ 2x \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 1 = 3 - 4\lambda + \lambda^2 + 1 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$$

$$\lambda_{1,2} = 2$$

$$\lambda_1 = 2: \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \vec{v}_1 = \vec{0}$$

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \sim (1 \ 1) \Rightarrow x+y=0$$

$$y = t, t \in \mathbb{R} - \{0\} \Rightarrow \vec{v}_1 = t \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$x = -y = -t \quad t=1 \cdot \vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 2: (A - \lambda E)^2 = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$0 \cdot x + 0 \cdot y = 0 \Rightarrow x, y \text{ cokohlik, aby nebyl}$$

$$\text{na sobrem } \vec{v}_1: \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\vec{y}_1 = e^{2x} \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \vec{y}_2 = e^{2x} [E + x(A - \lambda E)] \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = e^{2x} \begin{pmatrix} 1-x & -x \\ x & x+1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= e^{2x} \begin{pmatrix} 1-2x \\ 2x+1 \end{pmatrix}$$

$$\vec{y}_H = c_1 e^{2x} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{2x} \begin{pmatrix} 1-2x \\ 2x+1 \end{pmatrix}$$

$$\vec{b}(x) = \begin{pmatrix} -x^2 \\ 2x \end{pmatrix} = e^{0x} \begin{pmatrix} -x^2 \\ 2x \end{pmatrix} \dots \text{je 0 rovenem char.}$$

polynom matice soustavy? $\frac{NE}{\Delta}$ a

střihy jsou polynomy stupně nejvýše 2 $\Rightarrow \vec{y}_p = \begin{pmatrix} Ax^2 + Bx + C \\ Dx^2 + Ex + F \end{pmatrix}$

$$\vec{y}_p = \begin{pmatrix} 2Ax + B \\ 2Dx + E \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} 2Ax + B \\ 2Dx + E \end{pmatrix} &= \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} Ax^2 + Bx + C \\ Dx^2 + Ex + F \end{pmatrix} + \begin{pmatrix} -x^2 \\ 2x \end{pmatrix} = \\ &= \begin{pmatrix} Ax^2 + Bx + C - Dx^2 - Ex - F \\ Ax^2 + Bx + C + 3Dx^2 + 3Ex + 3F \end{pmatrix} + \begin{pmatrix} -x^2 \\ 2x \end{pmatrix} = \\ &= \begin{pmatrix} Ax^2 + Bx + C - Dx^2 - Ex - F - x^2 \\ Ax^2 + Bx + C + 3Dx^2 + 3Ex + 3F + 2x \end{pmatrix} \Rightarrow \end{aligned}$$

$$\Rightarrow \begin{aligned} 1) \quad 2Ax + B &= Ax^2 + Bx + C - Dx^2 - Ex - F - x^2 \\ 2) \quad 2Dx + E &= Ax^2 + Bx + C + 3Dx^2 + 3Ex + 3F + 2x \end{aligned}$$

$$\begin{aligned} 1) \quad x^2: \quad 0 &= A - D - 1 \\ x^1: \quad 2A &= B - E \\ x^0: \quad B &= C - F \end{aligned}$$

$$\begin{aligned} 2) \quad x^2: \quad 0 &= A + 3D \\ x^1: \quad 2D &= B + 3E + 2 \\ x^0: \quad E &= C + 3F \end{aligned}$$

soustava 6 rovnic o 6 neznámých
její řešení je:

$$\begin{aligned} A &= \frac{3}{4} \\ B &= \frac{1}{2} \\ C &= \frac{1}{8} \\ D &= -\frac{1}{4} \\ E &= -1 \\ F &= -\frac{3}{8} \end{aligned}$$

$$\vec{y} = \vec{y}_H + \vec{y}_p = \underline{c_1 e^{2x} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{2x} \begin{pmatrix} -2x+1 \\ 2x+1 \end{pmatrix} + \begin{pmatrix} \frac{3}{4}x^2 + \frac{1}{2}x + \frac{1}{8} \\ -\frac{1}{4}x^2 - x - \frac{3}{8} \end{pmatrix}}$$

$$c_1, c_2 \in \mathbb{R}$$

$$c) \vec{y}' = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \vec{y} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-x}$$

$$\begin{vmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda+1)(\lambda-3) = 0$$

$$\Leftrightarrow \lambda_1 = -1, \lambda_2 = 3$$

$$\underline{\lambda_1 = -1:}$$

$$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \vec{v}_1 = \vec{0}$$

$$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \stackrel{!}{\sim} \begin{pmatrix} 1 & 2 \\ \cancel{1} & \cancel{2} \end{pmatrix} \sim (1 \ 2)$$

$$\Rightarrow x + 2y = 0$$

$$y = t$$

$$x = -2y = -2t, t \in \mathbb{R} - \{0\}$$

$$\vec{v}_1 = \begin{pmatrix} -2t \\ t \end{pmatrix} = t \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$t = 1: \vec{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\underline{\lambda_2 = 3:}$$

$$\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} \vec{v}_2 = \vec{0}$$

$$\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} \stackrel{!}{\sim} \begin{pmatrix} 1 & -2 \\ \cancel{1} & \cancel{-2} \end{pmatrix} \sim (1 \ -2) \Rightarrow x - 2y = 0$$

$$y = t$$

$$x = 2y = 2t, t \in \mathbb{R} - \{0\}$$

$$\vec{v}_2 = \begin{pmatrix} 2t \\ t \end{pmatrix} = t \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$t = 1: \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\vec{y}_1 = e^{-x} \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \vec{y}_2 = e^{3x} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\underline{\vec{y}_H = c_1 e^{-x} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 e^{3x} \begin{pmatrix} 2 \\ 1 \end{pmatrix}} \quad | c_1, c_2 \in \mathbb{R}$$

$$\vec{b}(x) = e^{-x} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

je -1 eigenem char. polynomu matice soustavy?

ANO, jednonásobný roven

a složky vektoru $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ jsou polynomy

stupně nejvýše 0 $\Rightarrow \vec{y}_p = e^{-x} \cdot \begin{pmatrix} Ax+B \\ Cx+D \end{pmatrix}$ \leftarrow polynomy
 stupně $(0+1)=1$

$$\vec{y}_p = \begin{pmatrix} e^{-x}(Ax+B) \\ e^{-x}(Cx+D) \end{pmatrix}$$

$$\vec{y}_p' = \begin{pmatrix} -e^{-x}(Ax+B) + e^{-x} \cdot A \\ -e^{-x}(Cx+D) + e^{-x} \cdot C \end{pmatrix} = e^{-x} \begin{pmatrix} -(Ax+B) + A \\ -(Cx+D) + C \end{pmatrix}$$

$$e^{-x} \begin{pmatrix} -(Ax+B) + A \\ -(Cx+D) + C \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \cdot e^{-x} \begin{pmatrix} Ax+B \\ Cx+D \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-x} \quad | : e^{-x}$$

$$\begin{pmatrix} -Ax - B + A \\ -Cx - D + C \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} Ax+B \\ Cx+D \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -Ax - B + A \\ -Cx - D + C \end{pmatrix} = \begin{pmatrix} Ax+B + 4Cx + 4D + 1 \\ Ax+B + Cx + D + 1 \end{pmatrix}$$

$$1) -Ax - B + A = Ax + B + 4Cx + 4D + 1$$

$$2) -Cx - D + C = Ax + B + Cx + D + 1$$

$$1) x^1: -A = A + 4C$$

$$x^0: -B + A = B + 4D + 1$$

$$2) x^1: -C = A + C$$

$$x^0: -D + C = B + D + 1$$

$$0 = -2A + 4C \dots \text{2 masovek me} \rightarrow \text{skritum}$$

$$-1 = 2B - A + 4D$$

$$0 = A + 2C \leftarrow$$

$$-1 = B - C + 2D$$

$$-1 = 2B - A + 4D \leftarrow$$

$$0 = A + 2C \Rightarrow A = -2C$$

$$-1 = B - C + 2D$$

$$-1 = 2B + 2C + 4D \leftarrow$$

$$-1 = B - C + 2D \Rightarrow B = -1 + C - 2D$$

$$-1 = -2 + 2C - 4D + 2C + 4D$$

$$1 = +4C \Rightarrow \boxed{C = \frac{1}{4}}, \boxed{A} = -2 \cdot \frac{1}{4} = \boxed{-\frac{1}{2}}, B = -1 + \frac{1}{4} - 2D =$$

$$= -\frac{3}{4} - 2D$$

$$\boxed{D = t}, t \in \mathbb{R}$$

$$\boxed{B = -\frac{3}{4} - 2t}$$

$$\Rightarrow \vec{y}_p = e^{-x} \begin{pmatrix} -\frac{1}{2}x - \frac{3}{4} - 2t \\ \frac{1}{4}x + t \end{pmatrix}, t \in \mathbb{R}$$

volime $t = 0: \vec{y}_p = e^{-x} \begin{pmatrix} -\frac{x}{2} - \frac{3}{4} \\ \frac{x}{4} \end{pmatrix}$

$$\underline{\underline{\vec{y} = c_1 e^{-x} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 e^{3x} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + e^{-x} \begin{pmatrix} -\frac{x}{2} - \frac{3}{4} \\ \frac{x}{4} \end{pmatrix}, c_1, c_2 \in \mathbb{R}}}$$

