KŘIVKOVÝ INTEGRÁL 1. DRUHU

po částech hladká křivka C zadaná rovnicí

$$r(s) = x(s)\cdot i + y(s)\cdot j\ , \quad 0 \le s \le a$$

v každém bodě křivky C je zadána spojitá funkce f

$$\int_C f(x,y)\,ds = \int_0^a f(x(s), y(s))\,ds = \int_a^b f(x(t), y(t))\sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt$$

PŘÍKLAD

Vypočtěte:

a) $\int_C (x^2 - y + 3z)\,ds$, kde C je úsečka s počátečním bodem $[0,0,0]$ a koncovým bodem $[1,2,1]$

$$C: \quad X = A + \vec{u}\cdot t = A + (B-A)\cdot t = [0,0,0] + (1,2,1)\cdot t$$

$$\left.\begin{aligned} x(t) &= t \\ y(t) &= 2t \\ z(t) &= t \end{aligned}\right\} \Rightarrow r(t) = t\cdot i + 2t\cdot j + t\cdot k, \quad t \in \langle 0, 1\rangle$$

$$\int_C (x^2 - y + 3z)\,ds = \int_0^1 (t^2 - 2t + 3t)\sqrt{1^2 + 2^2 + 1^2}\,dt = \int_0^1 (t^2 + t)\sqrt{6}\,dt =$$

$$= \sqrt{6}\left[\frac{t^3}{3} + \frac{t^2}{2}\right]_0^1 = \sqrt{6}\cdot\left(\frac{1}{3} + \frac{1}{2}\right) = \sqrt{6}\cdot\frac{5}{6}$$

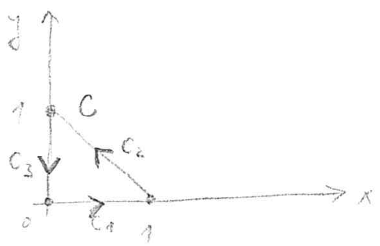
b) $\int_C (x^2 + y^2)\,ds$, kde C je úsečka spojující body $[0,0]$, $[1,1]$

$$C: X = [0,0] + (1,1)t$$

$$\left.\begin{aligned} x(t) &= t \\ y(t) &= t \end{aligned}\right\} \Rightarrow r(t) = t\cdot i + t\cdot j\ , \quad t \in \langle 0, 1\rangle$$

$$\int_C (x^2 + y^2)\,ds = \int_0^1 (t^2 + t^2)\sqrt{1^2 + 1^2}\,dt = 2\int_0^1 t^2\sqrt{2}\,dt = 2\sqrt{2}\left[\frac{t^3}{3}\right]_0^1 = \frac{2\sqrt{2}}{3}$$

c) $\int_C (x^2 + y^2)\,ds$, kde C je hranice trojúhelníku s vrcholy $[0,0]$, $[1,0]$, $[0,1]$, kladně orientovaná



$$C = C_1 \cup C_2 \cup C_3$$

$$C_1: X = [0,0] + (1,0)t$$

$$\begin{aligned} x(t) &= t \\ y(t) &= 0 \end{aligned}\ , \quad t \in \langle 0, 1\rangle$$

$(1,0) - (1,1) = (0, \ldots$

$$C_2: X = [1,0] + (-1,1)t$$

$$\begin{aligned} x(t) &= 1 - t \\ y(t) &= t \end{aligned}\ , \quad t \in \langle 0, 1\rangle$$

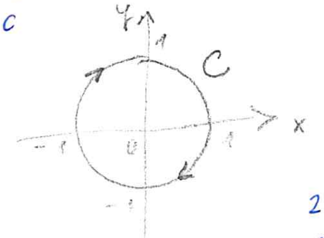
(1)

$$C_2^*: \begin{aligned} x &= 0 + 0 \\ y &= 1 - t \end{aligned}$$

$$C_3:\ X = [0,1] + (0,-1)t$$
$$x(t) = 0,\quad y(t) = 1-t,\quad t \in \langle 0, 1\rangle$$

$$\int_C (x^2+y^2)\,ds = \int_0^1 t^2\cdot\sqrt{1^2+0^2}\,dt + \int_0^1 [\underbrace{(1-t)^2}_{1-2t+t^2} + t^2]\sqrt{1^2+1^2}\,dt +$$
$$+ \int_0^1 (1-t)^2\cdot\sqrt{0^2+1^2}\,dt = \left[\frac{t^3}{3}\right]_0^1 + \left[t - 2\frac{t^2}{2} + 2\frac{t^3}{3}\right]_0^1\sqrt{2} +$$
$$+ \left[t - 2\frac{t^2}{2} + \frac{t^3}{3}\right]_0^1 = \frac{1}{3} + \sqrt{2}\left(1 - 1 + \frac{2}{3}\right) + \left(1 - 1 + \frac{1}{3}\right) =$$
$$= \frac{1}{3} + \frac{2\sqrt{2}}{3} + \frac{1}{3} = \frac{2(1+\sqrt{2})}{3}$$

d) $\int_C (x^2+y^2)\,ds$, kde C je kružnice $x^2+y^2=1$ záporně orientovaná



$$C:\ x(t) = r\cdot\cos t = \cos t,\quad y(t) = r\cdot\sin t = \sin t,\quad t\in\langle 0, 2\pi\rangle$$

$$\int_C (x^2+y^2)\,ds = \int_0^{2\pi} (\underbrace{\cos^2 t + \sin^2 t}_{1})\cdot\underbrace{\sqrt{(-\sin t)^2 + \cos^2 t}}_{1}\,dt = [t]_0^{2\pi} = 2\pi$$

e) $\int_C (y^2-x^2)\,ds$, kde C je zadána parametrickými rovnicemi $x(t) = e^{-t}$, $y(t) = e^{t}$, $t\in\langle -1, 1\rangle$

$$\int_C (y^2-x^2)\,ds = \int_{-1}^1 [(e^t)^2 - (e^{-t})^2]\sqrt{(-e^{-t})^2 + (e^t)^2}\,dt = \frac{1}{2}\int_{-1}^1 2(e^{2t} - e^{-2t})\cdot$$
$$\cdot\sqrt{e^{-2t}+e^{2t}}\,dt = \left| \begin{array}{l} u = e^{-2t} + e^{2t} \\ du = (-2e^{-2t} + 2e^{2t})\,dt = 2(e^{2t} - e^{-2t})\,dt \\ t = -1 \rightarrow u = e^2 + e^{-2} \\ t = 1 \rightarrow u = e^2 + e^{-2} \end{array} \right| =$$

$$= \frac{1}{2}\int_{e^2+e^{-2}}^{e^2+e^{-2}} u^{1/2}\,du = \frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right]_{e^2+e^{-2}}^{e^2+e^{-2}} = \frac{1}{3}\left[u\sqrt{u}\right]_{e^2+e^{-2}}^{e^2+e^{-2}} \overset{\text{stejné meze}}{=\!=} 0$$

~~f)~~ $\int_C x\,ds$, kde C je zadána parametrickými rovnicemi $x(t) = a(1-\cos t)$, $y(t) = a(t - \sin t)$, $t\in\langle 0, 2\pi\rangle$

$$\int_C x\,ds = \int_0^{2\pi} a\cdot(1-\cos t)\cdot\sqrt{(+a\cdot\sin t)^2 + (a - a\cdot\cos t)^2}\,dt =$$
$$= \int_0^{2\pi} a(1-\cos t)\sqrt{a^2\cdot\sin^2 t + a^2 - 2a^2\cos t + a^2\cos^2 t}\,dt =$$

a^2

$$= \int_0^{2\pi} a^2 (1-\cos t)\sqrt{2}\cdot\sqrt{1-\cos t}\, dt = a^2\sqrt{2}\int_0^{2\pi} (1-\cos t)^{3/2}\, dt = a^2\sqrt{2}\cdot \frac{16\sqrt{2}}{3} =$$

$$= \frac{32}{3}a^2$$

KŘIVKOVÝ INTEGRÁL 2. DRUHU

po částech hladká křivka C zadaná rovnicí

$$r(s) = x(s)\cdot i + y(s)\cdot j\,,\quad 0 \le s \le \lambda$$

vektorové pole na otevřené mn. M

$$a(x,y) = u(x,y)\cdot i + v(x,y)\cdot j$$

$$\int_0^\lambda a\cdot t\, ds = \int_C a\, dr = \int_C u\, dx + v\, dy$$

křivkový integrál 2. druhu vyjadřuje práci vektoru a po křivce C

je-li C^* opačně orientovaná k C, pak

$$\int_{C^*} a\, dr = \int_0^\lambda a\cdot(-t)\, ds = -\int_0^\lambda a\cdot t\, ds = -\int_C a\, dr$$

jednotkový tečný vektor: $t = \dfrac{r'(t)}{|r'(t)|}$

$$\int_C a\, dr = \int_a^b a(x(t), y(t))\, r'(t)\, dt = \int_a^b [u(x(t),y(t))\cdot i + v(x(t),y(t))\cdot j]\cdot[x'(t)\cdot i + y'(t)\cdot j]\, dt =$$

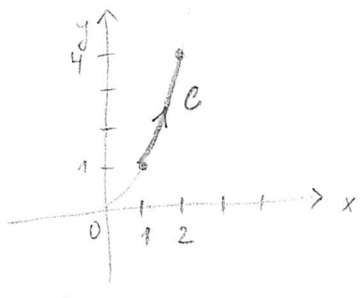
$$= \int_a^b [u(x(t),y(t))\cdot x'(t) + v(x(t),y(t))\cdot y'(t)]\, dt =$$

$$= \int_C u\, dx + v\, dy$$

PŘÍKLAD

Vypočtěte:

a) $\int_C dx + x\, dy$, kde C je část paraboly $y = x^2$ s počátečním bodem $[1,1]$ a koncovým bodem $[2,4]$



$$a(x,y) = u(x,y)\cdot i + v(x,y)\cdot j = 1\cdot i + x\cdot j = i + x\cdot j = (1; x)$$

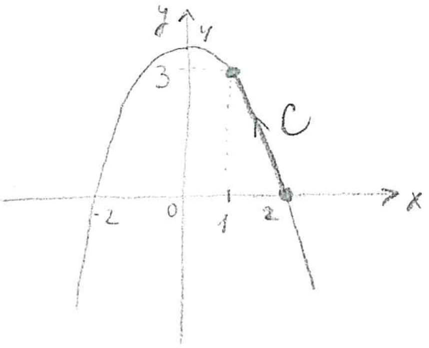
$$C:\ x(t) = t\,,\ y(t) = t^2\,,\ t \in \langle 1, 2\rangle$$

$$\int_C dx + x\, dy = \int_1^2 (1\cdot 1 + t\cdot 2t)\, dt = \int_1^2 (1 + 2t^2)\, dt = \left[\, t + 2\frac{t^3}{3}\right]_1^2 =$$

$$= \left(2 + \frac{16}{3}\right) - \left(1 + \frac{2}{3}\right) = 1 + \frac{14}{3} = \frac{17}{3}$$

b) $\int_C y\,dx + x^2\,dy$, kde C je část paraboly $y = 4-x$ od bodu $[2,0]$ do bodu $[1,3]$

$C: x(t) = t,\ y(t) = 4-t^2,\ t \in \langle 2, 1\rangle$

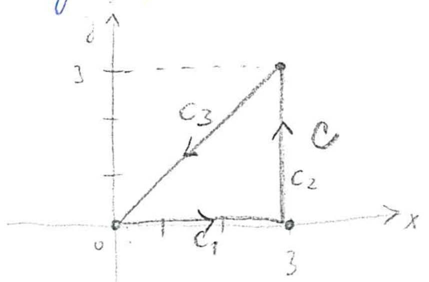


$$\int_C y\,dx + x^2\,dy = -\int_1^2 \left[(4-t^2)\cdot 1 + t^2\cdot(-2t)\right] dt = -\int_1^2 (4-t^2-2t^3)\,dt =$$

$$= -\left[4t - \frac{t^3}{3} - 2\frac{t^4}{4}\right]_1^2 = -\left[\left(\cancel{8} - \frac{8}{3} - \cancel{\frac{16}{2}}\right) - \left(4 - \frac{1}{3} - \frac{1}{2}\right)\right] = -\left(-\frac{8}{3} - 4 + \frac{1}{3} + \frac{1}{2}\right) =$$

$$= \frac{8}{3} + 4 - \frac{1}{3} - \frac{1}{2} = \frac{7}{3} - \frac{1}{2} + 4 = \frac{14-3+24}{6} = \frac{35}{6}$$

c) $\int_C (2x-y)\,dx + (x+3y)\,dy$, kde C je hranice trojúhelníku s vrcholy $[0,0]$, $[3,0]$, $[3,3]$



$C = C_1 \cup C_2 \cup C_3$

$C_1: x(t) = 3t,\ y(t) = 0,\ t \in \langle 0,1\rangle$

$C_2: x(t) = 3,\ y(t) = 3t,\ t \in \langle 0,1\rangle$

$C_3: x(t) = 3-3t,\ y(t) = 3-3t,\ t \in \langle 0,1\rangle$

$$\int_C (2x-y)\,dx + (x+3y)\,dy = \int_0^1 \left[6t\cdot 3 + 3t\cdot 0\right] dt + \int_0^1 \left[(6-3t)\cdot 0 + (3+9t)\cdot 3\right] dt +$$

$$+ \int_0^1 \left[(3-3t)\cdot(-3) + (12-12t)\cdot(-3)\right] dt = \int_0^1 18t\,dt + \int_0^1 9\cdot(1+3t)\,dt + 45\int_0^1 (t-1)\,dt =$$

$$= \overset{9}{\cancel{18}}\left[\frac{t^2}{\cancel{2}}\right]_0^1 + 9\left[t + \frac{3t^2}{2}\right]_0^1 + 45\left[\frac{t^2}{2} - t\right]_0^1 = 9 + 9\left(1 + \frac{3}{2}\right) + 45\left(\frac{1}{2} - 1\right) = 9 + 9 + \frac{27}{2} -$$

$$- \frac{45}{2} = 18 - \frac{18}{2} = 9$$

d) $\int_C a\,dr$, kde $a = xy\cdot i + y\cdot j$, $C: r(t) = 4t\cdot i + t\cdot j$; $t \in \langle 0, 1\rangle$

$u(x,y) = xy$, $v(x,y) = y$, $x(t) = 4t$, $y(t) = t$

$$\int_C a\,dr = \int_0^1 (4t^2, t)\cdot(4,1)\,dt = \int_0^1 (16t^2 + t)\,dt = \left[16\frac{t^3}{3} + \frac{t^2}{2}\right]_0^1 =$$

$$= \frac{16}{3} + \frac{1}{2} = \frac{32+3}{6} = \frac{35}{6}$$

e) $\int_C a\,dr$ $= 3x\cdot i + 4y\cdot j$, $C: r(t) = 2\cos t\cdot i + 2\sin t\cdot j$, $t \in \langle 0, \pi/2\rangle$

$u(x,y) = 3x$, $v(x,y) = 4y$, $x(t) = 2\cos t$, $y(t) = 2\sin t$

$$\int_C a\,dr = \int_0^{\pi/2} (6\cos t, 8\sin t)\cdot(-2\sin t, 2\cos t)\,dt = \int_0^{\pi/2} (-12\sin t\cos t +$$

$$+ 16\sin t\cos t)\,dt = \int_0^{\pi/2} 4\sin t\cdot\cos t\,dt = \left|\begin{array}{l} u = \sin t \\ du = \cos t\,dt \\ t = 0 \to u = 0 \\ t = \pi/2 \to u = 1 \end{array}\right| =$$

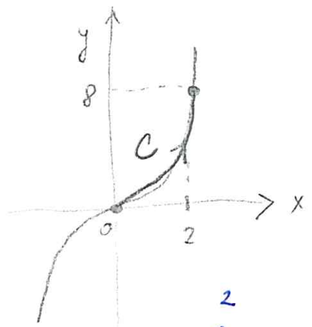
$$= 4\int_0^1 u\,du = 4\left[\frac{u^2}{2}\right]_0^1 = 2$$

PŘÍKLAD

Vypočtěte práci vektoru a po křivce C zadaných následovně:

Vypočtěte práci vektoru a po části kubické paraboly $y = x^3$

a) $a(x,y) = -x\cdot i - 2y\cdot j$, C je část kubické paraboly $y = x^3$ od bodu $[0,0]$ do bodu $[2,8]$



$$C: \begin{array}{l} x(t) = t \\ y(t) = t^3 \end{array}, \quad t \in \langle 0, 2\rangle$$

$$\int_C a\,dr = \int_0^2 (-t, -2t^3)\cdot(1, 3t^2)\,dt = \int_0^2 (-t - 6t^5)\,dt = \left[-\frac{t^2}{2} - 6\frac{t^6}{6}\right]_0^2 =$$

$$= -2 - 64 = -66$$

b) $a = x\cdot i + y\cdot j - 5z\cdot k$, $C: r(t) = 2\cos t\cdot i + 2\sin t\cdot j + t\cdot k$, $t \in \langle 0, 2\pi\rangle$

$$\int_C a\,dr = \int_0^{2\pi} (2\cos t, 2\sin t, -5t)\cdot(-2\sin t, 2\cos t, 1)\,dt =$$

$$= \int_0^{2\pi} (-4\sin t\cdot\cos t + 4\sin t\cdot\cos t - 5t)\,dt = \int_0^{2\pi} -5t\,dt = -5\left[\frac{t^2}{2}\right]_0^{2\pi} =$$

$$= -\frac{5}{2}\cdot 4\pi^2 = -10\pi^2$$

NEZÁVISLOST NA INTEGRAČNÍ CESTĚ

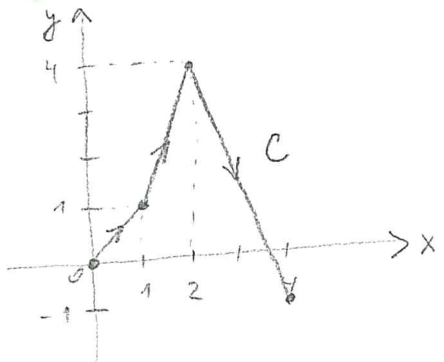
$\int_C \boldsymbol{a}\, d\boldsymbol{r}$ ... je-li $\boldsymbol{a}$ potenciálové pole, pak $\int_C \boldsymbol{a}\, d\boldsymbol{r}$ nezávisí na tom, na které křivce se integruje, závisí pouze na jejím počátečním a koncovém bodě:

$$\int_C \boldsymbol{a}\, d\boldsymbol{r} = f(x(b), y(b)) - f(x(a), y(a)) = \left[f(x,y) \right]_{[x(a),\, y(a)]}^{[x(b),\, y(b)]}$$

kde f je potenciálová funkce vektorového pole (analogicky pro R^3)

PŘÍKLAD

Vypočtěte $\int_C y\, dx + x\, dy$, kde C je lomená čára spojující body $[0,0]$, $[1,1]$, $[2,4]$, $[4,-1]$.



$\boldsymbol{a}(x,y) = y \cdot \boldsymbol{i} + x \cdot \boldsymbol{j}$

$u'_y = 1, \; v'_x = 1 \Rightarrow \boldsymbol{a}$ potenciálové pole

Jak vypadá potenciálová fce f?

1) $f'_x = u = y \;\wedge\; f'_y = v = x$

2) $f(x,y)\, dx = \int f'_x\, dx = \int y\, dx = xy + g(y) + k_1 \;\wedge$

$f(x,y)\, dy = \int f'_y\, dy = \int x\, dy = xy + h(x) + k_2$

$xy + g(y) + k_1 = xy + h(x) + k_2$

$g(y) = h(x) \Rightarrow g(y) = h(x) = 0$

$\Rightarrow f(x,y) = xy + k$

$$\int_C y\, dx + x\, dy = \left[xy + k \right]_{[0,0]}^{[4,-1]} = -4 - 0 = -4$$

PŘÍKLAD

Vypočtěte:

a) $\int_C 2xy\,dx + (x^2-y)\,dy$, kde C je libovolná hladká křivka s počátečním bodem $[-1,4]$ a koncovým $[1,2]$

$a(x,y) = (2xy,\ x^2-y)$

$u_y' = 2x = v_x' \Rightarrow a$ je potenciálové pole

nalezení potenciálové funkce:

$f(x,y) = \int f_x'\,dx = \int u\,dx = \int 2xy\,dx = 2y\frac{x^2}{2} + g(y) + k_1$

$f(x,y) = \int f_y'\,dy = \int v\,dy = \int (x^2-y)\,dy = x^2y - \frac{y^2}{2} + h(x) + k_2$

$x^2y + g(y) + k_1 = x^2y - \frac{y^2}{2} + h(x) + k_2 \Rightarrow g(y) = \frac{-y^2}{2},\ h(x) = 0$

$\Rightarrow f(x,y) = x^2y - \frac{y^2}{2} + k$

$$\int_C 2xy\,dx + (x^2-y)\,dy = \left[x^2y - \frac{y^2}{2}\right]_{[-1,4]}^{[1,2]} = (2-2) - (4-8) = 4$$

b) $\int_C e^x\cdot\sin y\,dx + e^x\cdot\cos y\,dy$, kde C je část cykloidy $x = t - \sin t$, $y = 1-\cos t$ od bodu $[0,0]$ do bodu $[2\pi, 0]$

$a(x,y) = (e^x\cdot\sin y,\ e^x\cdot\cos y)$

$u_y' = e^x\cdot\cos y = v_x' \Rightarrow a$ je potenciálové pole

nalezení potenciálové funkce:

$f(x,y) = \int f_x'\,dx = \int e^x\cdot\sin y\,dx = e^x\cdot\sin y + g(y) + k_1$

$f(x,y) = \int f_y'\,dy = \int e^x\cdot\cos y\,dy = e^x\cdot\sin y + h(x) + k_2$

$e^x\cdot\sin y + g(y) + k_1 = e^x\cdot\sin y + h(x) + k_2$

$g(y) = h(x) = 0$

$\Rightarrow f(x,y) = e^x\cdot\sin y + k$

$$\int_C e^x\cdot\sin y\,dx + e^x\cdot\cos y\,dy = \left[e^x\cdot\sin y\right]_{[0,0]}^{[2\pi,0]} = e^{2\pi}\cdot 0 - e^0\cdot 0 = 0$$

GREENOVA VETA

popisuje vzťah medzi krivkovým integrálom 2. druhu a dvojným integrálom

Uvažujme množinu $M \subset \mathbb{R}^2$, ktorej hranicu tvorí jednoduchá uzavretá po častiach hladká krivka C. Nech u, v, u'_y, v'_x sú spojité na M i C. Potom platí

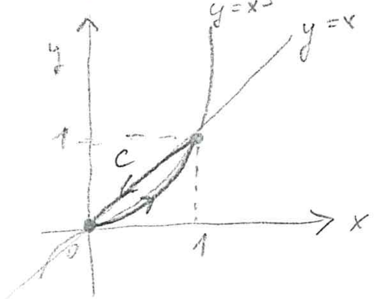
$$\int_C u(x,y)\,dx + v(x,y)\,dy = \iint_M [v'_x(x,y) - u'_y(x,y)]\,dx\,dy,$$

kde C je orientovaná kladne.

PŘÍKLAD

Vypočítajte:

a) $\int_C y^3\,dx + (x^3+3xy^2)\,dy$, kde C je kladne orientovaná uzavretá krivka, skladajúca sa z časti kubickej paraboly $y = x^3$ od bodu $[0,0]$ do bodu $[1,1]$ a úsečky $y = x$ spájajúcej body $[0,0]$ a $[1,1]$



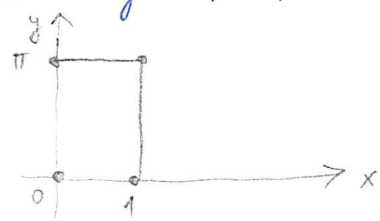
$M: \quad 0 \le x \le 1$
$\quad\quad x^3 \le y \le x$

$$\int_C \underbrace{y^3}_{u}\,dx + \underbrace{(x^3+3xy^2)}_{v}\,dy = \iint_M (3x^2+3y^2-3y^2)\,dx\,dy =$$

$$= \int_0^1 \int_{x^3}^{x} 3x^2\,dy\,dx = \int_0^1 3x^2 [y]_{x^3}^{x}\,dx = \int_0^1 3x^2(x-x^3)\,dx =$$

$$= 3\int_0^1 (x^3-x^5)\,dx = 3\left[\frac{x^4}{4} - \frac{x^6}{6}\right]_0^1 = 3\left(\frac{1}{4}-\frac{1}{6}\right) = 3\cdot\frac{3-2}{12} = \frac{1}{4}$$

b) $\int_C e^x \cos y\,dx + e^x \sin y\,dy$, kde C je kladne orientovaná uzavretá krivka, ktorá je hranicou obdĺžnika s vrcholy $[0,0]$, $[1,0]$, $[1,\pi]$, $[0,\pi]$

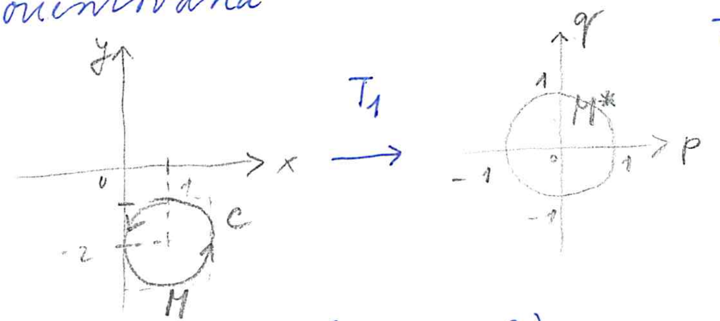


$M: \quad 0 \le x \le 1$
$\quad\quad 0 \le y \le \pi$

$$\int_C e^x \cos y\,dx + e^x \sin y\,dy = \iint_{0\,0} (e^x \sin y + e^x \sin y)\,dy\,dx =$$

$$= 2\int_0^1\int_0^\pi e^x \sin y\,dy\,dx = 2\left[e^x\right]_0^1 \cdot \left[-\cos y\right]_0^\pi = 2(e-1)(1+1) = 4(e-1)$$

~~d)~~ $\int_C (xy+3y^2)\,dx + (5xy+2x^2)\,dy$, kde $C: (x-1)^2+(y+2)^2=1$ je kladně orientovaná



T_1: transformace pro posunutí do počátku

$p = x-1 \Rightarrow x = p+1$

$q = y+2 \Rightarrow y = q-2$

$$|J| = 1 = \begin{vmatrix} \frac{\partial x}{\partial p} & \frac{\partial x}{\partial q} \\ \frac{\partial y}{\partial p} & \frac{\partial y}{\partial q} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$I = \int_C \underbrace{(xy+3y^2)}_{u'_y = x+6y}\,dx + \underbrace{(5xy+2x^2)}_{v'_x = 5y+4x}\,dy = \iint_M (5y+4x-x-6y)\,dx\,dy =$$

$$= \iint_M (3x-y)\,dx\,dy = \iint_{M^*} \left[3(p+1)-(q-2)\right] dp\,dq = \iint_{M^*} (3p-q+5)\,dp\,dq$$

T_2: transformace do pol. souřadnic

$p = \rho\cdot\cos\varphi \qquad 0 \le \varphi \le 2\pi$

$q = \rho\cdot\sin\varphi \qquad 0 \le \rho \le 1$

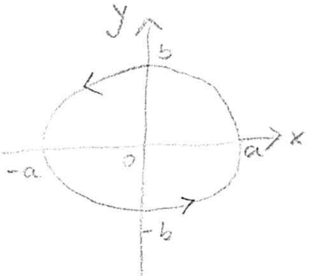
$|J| = \rho$

$$I = \int_0^{2\pi}\int_0^1 (3\rho\cos\varphi - \rho\sin\varphi + 5)\,\rho\,d\rho\,d\varphi = \int_0^{2\pi}\left[3\frac{\rho^3}{3}\cos\varphi - \frac{\rho^3}{3}\sin\varphi + 5\frac{\rho^2}{2}\right]_0^1 d\varphi =$$

$$= \int_0^{2\pi}\left[\rho^3\cos\varphi - \frac{\rho^3}{3}\sin\varphi + \frac{5}{2}\rho^2\right]_0^1 d\varphi = \int_0^{2\pi}\left(\cos\varphi - \frac{1}{3}\sin\varphi + \frac{5}{2}\right) d\varphi =$$

$$= \left[\sin\varphi + \frac{1}{3}\cos\varphi + \frac{5}{2}\varphi\right]_0^{2\pi} = \left(0 + \frac{1}{3}\cdot 1 + 5\pi\right) - \left(0 + \frac{1}{3}\cdot 1\right) = 5\pi$$

a) $\int_C (x+y)\,dx - x\,dy$, kde C je elipsa o poloosách $a, b > 0$, kladně orientovaná



$$= \int_C (x+y)\,dx - x\,dy = \iint_{M^*} (-1-1)\,dx\,dy = -2\iint_{M^*} dx\,dy$$

transformacia do pol. súradníc:

$$x = a\rho\cos\varphi \overset{\rho=1}{=} a\cos\varphi$$

$$y = b\rho\sin\varphi = b\cos\varphi$$

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} a\cos\varphi & -a\rho\sin\varphi \\ b\sin\varphi & b\rho\cos\varphi \end{vmatrix} = ab\rho\cos^2\varphi + ab\rho\sin^2\varphi =$$

$$= ab\rho$$

$$0 \le \varphi \le 2\pi$$

$$0 \le \rho \le \ ? : \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{a^2\rho^2\cos^2\varphi}{a^2} + \frac{b^2\rho^2\sin^2\varphi}{b^2} = 1$$

$$\rho^2 = 1 \Rightarrow 0 \le \rho \le 1$$

$$I = -2\int_0^{2\pi}\int_0^1 ab\rho\, d\rho\, d\varphi = -\cancel{2}ab\left[\frac{\rho^2}{\cancel{2}}\right]_0^1 [\varphi]_0^{2\pi} = -2\pi ab$$