TROJNÝ INTEGRÁL

Tělesem mezi grafy spojitých funkcí $z = F(x,y)$, $z = G(x,y)$, $[x,y] \in A$, kde $F(x,y) \leq G(x,y)$, nazýváme množinu B bodů $[x,y,z]$, pro něž platí nerovnosti

$$a \leq x \leq b$$
$$f(x) \leq y \leq g(x)$$
$$F(x,y) \leq z \leq G(x,y)$$

(A je kolmým průmětem mn. B do roviny xy)

PŘÍKLAD 1

Vyjádřete jako těleso mezi grafy spojitých funkcí

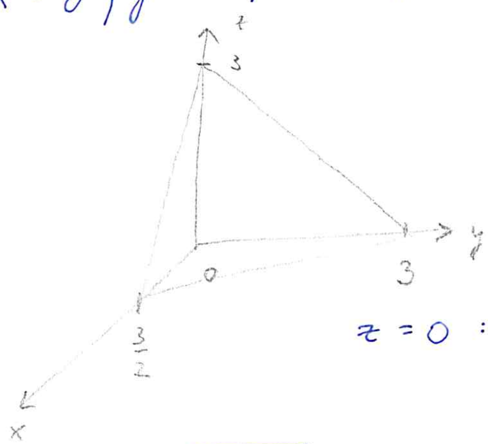
obor B ohraničený plochami:

a) $x=0,\ x=6,\ y=0,\ y=3,\ z=0,\ z=2$

$$\boxed{\begin{array}{l} 0 \leq x \leq 6 \\ 0 \leq y \leq 3 \\ 0 \leq z \leq 2 \end{array}}$$

b) $x=0,\ y=0,\ z=0,\ 2x+y+z=3$ $\to z = 3-2x-y$

$$\boxed{\begin{array}{l} 0 \leq x \leq \frac{3}{2} \\ 0 \leq y \leq 3-2x \\ 0 \leq z \leq 3-2x-y \end{array}}$$



$z=0: \ y = 3-2x$

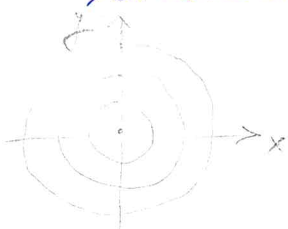
c) $z = \frac{h}{a}\sqrt{x^2+y^2}$, $z = h$, $a,h > 0$

řezy rovinami: $z = c,\ c > 0$: $c = \frac{h}{a}\sqrt{x^2+y^2}$

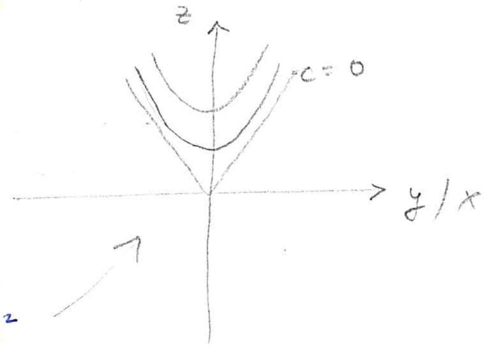
$$c^2 \cdot \left(\frac{a}{h}\right)^2 = x^2+y^2$$
$$x^2+y^2 = \left(\frac{c \cdot a}{h}\right)^2 = C \ \ldots \text{kružnice}$$

$c=0: \ [0,0]$

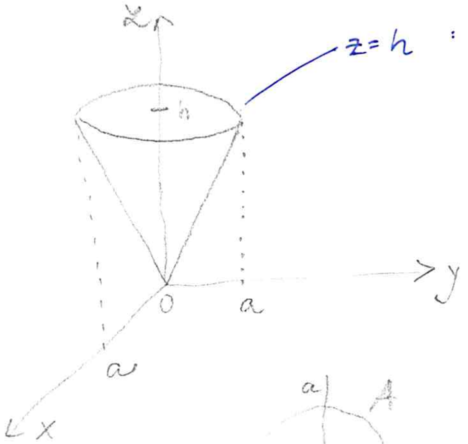
$c<0: \ \times$



$x = c: \quad z = \frac{h}{a}\sqrt{c^2+y^2}$



$y = c: \quad z = \frac{h}{a}\sqrt{x^2+c^2}$

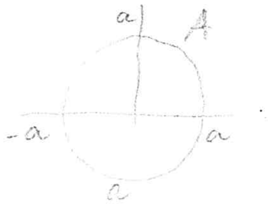


$z=h: \quad h = \frac{h}{a}\sqrt{x^2+y^2}$

$a^2 = x^2+y^2$

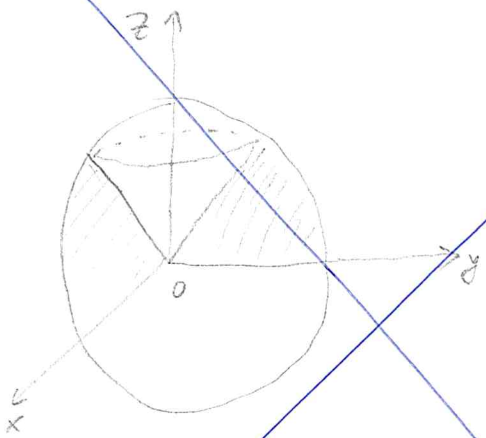
$y = \pm\sqrt{a^2-x^2}$

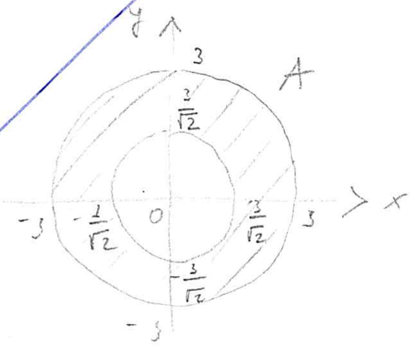
$$\boxed{\begin{aligned} -a &\le x \le a \\ -\sqrt{a^2-x^2} &\le y \le \sqrt{a^2-x^2} \\ \frac{h}{a}\sqrt{x^2+y^2} &\le z \le h \end{aligned}}$$



PŘÍKLAD 3

$x^2+y^2+z^2 = 9$ (uvnitř), $z = \sqrt{x^2+y^2}$ (vně), nad rovinou xy $(z=0)$





$x^2+y^2+x^2+y^2 = 9$

$2x^2+2y^2 = 9$

$x^2+y^2 = \frac{9}{2}$

Proveďte transformaci oboru B pomocí válcových souřadnic, je-li obor B ohraničený plochami:

a) jako v Příkladu 1c

$x = \rho\cos\varphi,\ y = \rho\sin\varphi,\ z = z$

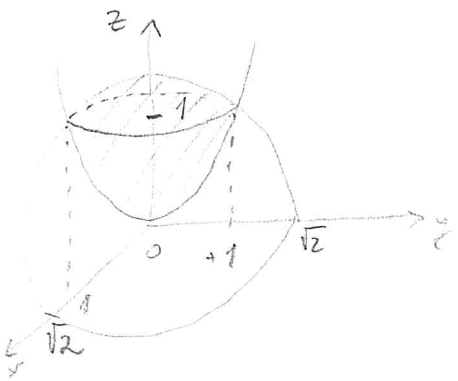
$$\boxed{\begin{aligned} &0 \le \varphi \le 2\pi \\ &0 \le \rho \le a \\ &\tfrac{h}{a}\cdot\sqrt{x^2+y^2} \le z \le h \\ &= \tfrac{h}{a}\cdot\rho \end{aligned}}$$

b) $x^2+y^2+z^2 = 2,\ z = x^2+y^2$

řezy rovinami: $z = c$: $c = x^2+y^2$ ... kružnice

$x = c$: $z = c^2+y^2$ ... parabola

$y = c$: $z = x^2+c^2$ ... —||—



$x^2+y^2+z^2 = 2 \wedge z = x^2+y^2$

$z + z^2 = 2$

$(z-1)(z+2) = 0$

$z = 1$ ✓

$\cancel{z = -2}$

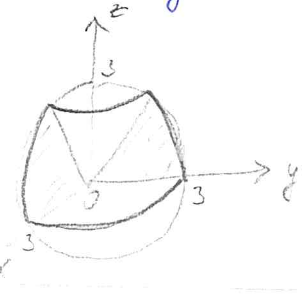
$z = 1:\ 1 = x^2+y^2$

$$\boxed{\begin{aligned} &0 \le \varphi \le 2\pi \\ &0 \le \rho \le 1 \\ &x^2+y^2 \le z \le \sqrt{2-x^2-y^2} = \\ &= \rho^2 \qquad\quad \sqrt{2-\rho^2} \end{aligned}}$$

PŘÍKLAD 4

Proveďte transformaci oboru B pomocí kulových souřadnic, je-li obor B ohraničený plochami:

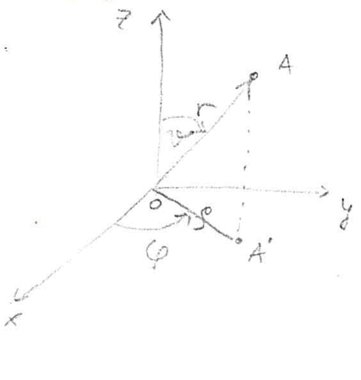
a) $x^2+y^2+z^2 = 9$ (uvnitř), $z = \sqrt{x^2+y^2}$ (vně), nad rovinou xy



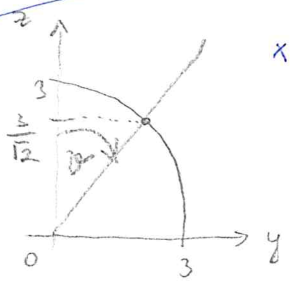
$x = r\cos\varphi\cdot\sin\vartheta$

$y = r\cdot\sin\varphi\cdot\sin\vartheta$

$z = r\cos\vartheta$



$$\boxed{\begin{aligned} 0 &\le r \le 3 \\ 0 &\le \varphi \le 2\pi \\ ? &\le \vartheta \le \frac{\pi}{2} \end{aligned}}$$



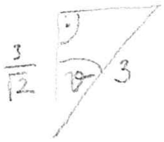
$x^2+y^2+z^2=9 \ \wedge \ z=\sqrt{x^2+y^2}$

$z^2+z^2=9$

$2z^2=9$

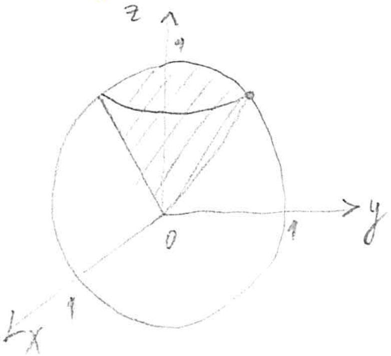
$z^2=\frac{9}{2}$

$z=\pm\frac{3}{\sqrt{2}}$



$\cos\vartheta = \frac{\frac{3}{\sqrt{2}}}{3} = \frac{\sqrt{2}}{2} \Rightarrow \vartheta = \frac{\pi}{4}$

b) $x^2+y^2+z^2=1$ (uvnitř), $z=\sqrt{x^2+y^2}$ (uvnitř)

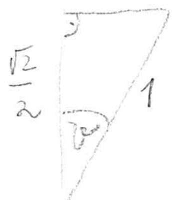


$$\boxed{\begin{aligned} 0 &\le r \le 1 \\ 0 &\le \varphi \le 2\pi \\ 0 &\le \vartheta \le ? \end{aligned}}$$

$x^2+y^2+z^2=1 \ \wedge \ z=\sqrt{x^2+y^2}$

$2z^2=1$

$z=\pm\frac{\sqrt{2}}{2}$



$\cos\vartheta = \frac{\sqrt{2}}{2} \Rightarrow \vartheta = \frac{\pi}{4}$

PŘÍKLAD 2.

Vypočtěte $\iiint_B f(x,y,z)\,dx\,dy\,dz$, je-li:

a) B jako v Př. 1a, $f(x,y,z)=x^2yz$

$$\iiint_B x^2yz\,dx\,dy\,dz = \int_0^2\int_0^3\int_0^6 x^2yz\,dx\,dy\,dz = \int_0^2\int_0^3 yz\left[\frac{x^3}{3}\right]_0^6 dy\,dz =$$

$$= 72\int_0^2 z\left[\frac{y^2}{2}\right]_0^3 dz = 72\int_0^2 \frac{9}{2}\,z\,dz = 324\left[\frac{z^2}{2}\right]_0^2 = 324\cdot 2 = 648$$

b) B jako v Př. [illegible], $g(x,y,z) = 2x - y -$ [illegible]

$$\iiint_B (2x-y-z)\,dx\,dy\,dz = \int_0^{\frac{3}{2}}\int_0^{3-2x}\int_0^{3-2x-y} (2x-y-z)\,dz\,dy\,dx =$$

$$= \int_0^{\frac{3}{2}}\int_0^{3-2x} \left[2xz - yz - \frac{z^2}{2}\right]_0^{3-2x-y} dy\,dx = \int_0^{\frac{3}{2}}\int_0^{3-2x} \left(-6x^2 - 2xy + 12x + \frac{y^2}{2} - \frac{9}{2}\right) dy\,dx$$

$$= \int_0^{\frac{3}{2}} \left[-6x^2y - 2x\frac{y^2}{2} + 12xy + \frac{y^3}{6} - \frac{9}{2}y\right]_0^{3-2x} dx = \int_0^{\frac{3}{2}} \frac{1}{3}(3-2x)^2(5x-3)\,dx =$$

$$= -\frac{27}{16}$$

PŘÍKLAD 5

Vypočtěte objem tělesa B, je-li:

a) B jako v Př. 1c

$$V = \iiint_B 1\,dx\,dy\,dz = \int_0^a\int_0^{2\pi}\int_{\frac{h}{a}\cdot\rho}^{h} \rho\,dz\,d\varphi\,d\rho = \int_0^a\int_0^{2\pi} \rho[z]_{\frac{h}{a}\cdot\rho}^{h}\,d\varphi\,d\rho =$$

$$= \int_0^a\int_0^{2\pi} \rho\left(h - \frac{h}{a}\cdot\rho\right) d\varphi\,d\rho = \int_0^a \left(\rho h - \frac{h}{a}\rho^2\right)[\varphi]_0^{2\pi}\,d\rho = 2\pi\left[h\frac{\rho^2}{2} - \frac{h}{a}\cdot\frac{\rho^3}{3}\right]_0^a$$

$$= 2\pi\left(\frac{ha^2}{2} - \frac{ha^3}{3a}\right) = 2\pi\left(\frac{ha^2}{2} - \frac{ha^2}{3}\right) = 2\pi\cdot\frac{3ha^2 - 2ha^2}{6} = \frac{\pi}{3}\cdot ha^2$$

b) B jako v Př. 4a

$$V = \iiint_B 1\,dx\,dy\,dz = \int_0^3\int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} r^2\cdot\sin\vartheta\,d\vartheta\,d\varphi\,dr = \int_0^3\int_0^{2\pi} r^2\cdot[-\cos\vartheta]_{\frac{\pi}{4}}^{\frac{\pi}{2}}\,d\varphi$$

$$dr = \int_0^3\int_0^{2\pi} r^2\left(0 + \frac{\sqrt{2}}{2}\right) d\varphi\,dr = \frac{\sqrt{2}}{2}\int_0^3 r^2[\varphi]_0^{2\pi}\,dr = \pi\sqrt{2}\left[\frac{r^3}{3}\right]_0^3 =$$

$$= 9\pi\sqrt{2}$$

PRIKLAD 6

Vypočítejte hmotnost a určete souřadnice těžiště tělesa B s hustotou $h(x,y,z)$, je-li:

a) B jako v Př. 3b, $h(x,y,z)=k$, $k>0$

$$m=\iiint_B h(x,y,z)\,dx\,dy\,dz=\int_0^1\int_0^{2\pi}\int_{\rho^2}^{\sqrt{2-\rho^2}} k\cdot\rho\,dz\,d\varphi\,d\rho=$$

$$=k\int_0^1\int_0^{2\pi}\rho\,[z]_{\rho^2}^{\sqrt{2-\rho^2}}\,d\varphi\,d\rho=k\int_0^1\int_0^{2\pi}\rho\left(\sqrt{2-\rho^2}-\rho^2\right)d\varphi\,d\rho=$$

$$=k\int_0^1\rho\left(\sqrt{2-\rho^2}-\rho^2\right)[\varphi]_0^{2\pi}\,d\rho=2k\pi\int_0^1\left(\rho\sqrt{2-\rho^2}-\rho^3\right)d\rho=$$

$$=\left|\begin{array}{l} t=2-\rho^2\\ dt=-2\rho\,d\rho\\ d\rho=\frac{dt}{-2\rho}\\ \rho=0\rightarrow t=2\\ \rho=1\rightarrow t=1\end{array}\right|=2k\pi\left(\int_2^1\rho\cdot t^{1/2}\cdot\frac{dt}{-2\rho}-\left[\frac{\rho^4}{4}\right]_0^1\right)=$$

$$=2k\pi\left(\frac{1}{2}\int_1^2 t^{1/2}\,dt-\frac{1}{4}\right)=k\pi\int_1^2 t^{1/2}\,dt-\frac{k\pi}{2}=k\pi\left[\frac{t^{3/2}}{3/2}\right]_1^2-\frac{k\pi}{2}=$$

$t\sqrt{t}$

$$=\frac{2}{3}k\pi\left(2\sqrt{2}-1\right)-\frac{k\pi}{2}=\frac{4k\pi\sqrt{2}}{3}-\frac{2}{3}k\pi-\frac{k}{2}\pi=\frac{4\sqrt{2}\pi k}{3}-\frac{4k\pi+3k\pi}{6}=$$

$$=\frac{4\sqrt{2}\pi k}{3}-\frac{7k\pi}{6}=\frac{k\pi}{6}\left(8\sqrt{2}-7\right)$$

$$x_T=y_T=0$$

$$z_T=\frac{6}{k\pi(8\sqrt{2}-7)}\int_0^1\int_0^{2\pi}\int_{\rho^2}^{\sqrt{2-\rho^2}} z\cdot k\cdot\rho\,dz\,d\varphi\,d\rho=\frac{6k}{k\pi(8\sqrt{2}-7)}\int_0^1\int_0^{2\pi}\rho\left[\frac{z^2}{2}\right]_{\rho^2}^{\sqrt{2-\rho^2}}d\varphi\,d\rho$$

$$=\frac{6}{2\pi(8\sqrt{2}-7)}\int_0^1\int_0^{2\pi}\left(2-\rho^2-\rho^4\right)\cdot\rho\,d\varphi\,d\rho=\frac{3}{\pi(8\sqrt{2}-7)}\cdot 2\pi\int_0^1\left(2\rho-\rho^3-\rho^5\right)d\rho=$$

$$=\frac{6}{8\sqrt{2}-7}\left[\rho^2-\frac{\rho^4}{4}-\frac{\rho^6}{6}\right]_0^1=\frac{6}{8\sqrt{2}-7}\cdot\left(1-\frac{1}{4}-\frac{1}{6}\right)=\frac{6}{8\sqrt{2}-7}\cdot\frac{12-3-2}{12}=\frac{7}{2(8\sqrt{2}-7)}$$

$$T=\left[0;0;\frac{7}{2(8\sqrt{2}-7)}\right]$$

b) B jako [illegible] $\sqrt{x^2+y^2+z^2}$

$$m = \iiint_B \sqrt{x^2+y^2+z^2}\, dx\,dy\,dz = \int_0^1\int_0^{2\pi}\int_0^{\frac{\pi}{4}} \sqrt{\underbrace{r^2\cos^2\varphi\sin^2 v + r^2\sin^2\varphi\sin^2 v}_{r^2\sin^2 v} + r^2\cos^2 v}\cdot r^2\sin v\, dv\,d\varphi\,dr = \int_0^1\int_0^{2\pi}\int_0^{\frac{\pi}{4}} r\cdot r^2\sin v\, dv\,d\varphi\,dr =$$

$$= \left[\frac{r^4}{4}\right]_0^1 \cdot \left[\varphi\right]_0^{2\pi}\cdot\left[-\cos v\right]_0^{\frac{\pi}{4}} = \frac{1}{4}\cdot 2\pi\cdot\left(-\frac{\sqrt{2}}{2}+1\right) = \frac{\pi}{2}\left(1-\frac{\sqrt{2}}{2}\right) =$$

$$= \frac{\pi}{4}\left(2-\sqrt{2}\right)$$

$$x_T = y_T = 0$$

$$z_T = \frac{4}{\pi(2-\sqrt{2})}\cdot\int_0^1\int_0^{2\pi}\int_0^{\frac{\pi}{4}} r\cdot\cos v\cdot r\cdot r^2\sin v\, dv\,d\varphi\,dr \overset{\frac{4}{\pi(2-\sqrt{2})}}{=} \left[\frac{r^5}{5}\right]_0^1\cdot\left[\varphi\right]_0^{2\pi}\cdot$$

$$\cdot\int_0^{\frac{\pi}{4}} \cos v\cdot\sin v\, dv = \frac{1}{5}\cdot 2\pi\cdot \overset{\frac{4}{\pi(2-\sqrt{2})}}{} \int_0^{\frac{\sqrt{2}}{2}} t\,dt = \frac{8}{5(2-\sqrt{2})}\cdot\left[\frac{t^2}{2}\right]_0^{\frac{\sqrt{2}}{2}} = \frac{4}{5(2-\sqrt{2})}\cdot\frac{2}{4} =$$

$$= \frac{2}{5(2-\sqrt{2})}$$

$$\left|\begin{array}{l} t = \sin v \\ dt = \cos v\,dv \\ v = 0 \to t = 0 \\ v = \frac{\pi}{4} \to t = \frac{\sqrt{2}}{2}\end{array}\right|$$

$$T = \left[0;\ 0;\ \frac{2}{5(2-\sqrt{2})}\right]$$

$$z_T = \frac{2}{\pi\left(1-\frac{\sqrt{2}}{2}\right)}\int_0^1\int_0^{2\pi}\int_0^{\frac{\pi}{4}} r\cdot\cos v\cdot r\cdot r^2\sin v\, dv\,d\varphi\,dr =$$

$$\left|\begin{array}{l} t = \cos v \\ dt = -\sin v\,dv \\ dv = \frac{dt}{-\sin v} \\ 0 \to 1 \\ \frac{\pi}{4} \to \frac{\sqrt{2}}{2}\end{array}\right|$$

$$= \frac{2}{\pi\left(1-\frac{\sqrt{2}}{2}\right)}\int_0^1\int_0^{2\pi}\int_1^{\frac{\sqrt{2}}{2}} r^4\cdot t\cdot(-dt)\,d\varphi\,dr = \frac{-2}{\pi\left(1-\frac{\sqrt{2}}{2}\right)}\int_0^1\int_0^{2\pi} r^4\left[\frac{t^2}{2}\right]_1^{\frac{\sqrt{2}}{2}} d\varphi\,dr =$$

$$= \frac{1}{\pi\left(\frac{\sqrt{2}}{2}-1\right)}\iint r^4\left(\frac{2}{4}-1\right)d\varphi\,dr = \frac{-1}{2\pi\left(\frac{\sqrt{2}}{2}-1\right)}\left[\varphi\right]_0^{2\pi}\cdot\left[\frac{r^5}{5}\right]_0^1 =$$

$$= \frac{1}{\cancel{2\pi}\left(1-\frac{\sqrt{2}}{2}\right)}\cdot\cancel{2\pi}\cdot\frac{1}{5} = \frac{1}{5\left(1-\frac{\sqrt{2}}{2}\right)} = \frac{1}{\frac{5}{2}\left(2-\sqrt{2}\right)} = \frac{2}{5(2-\sqrt{2})}$$