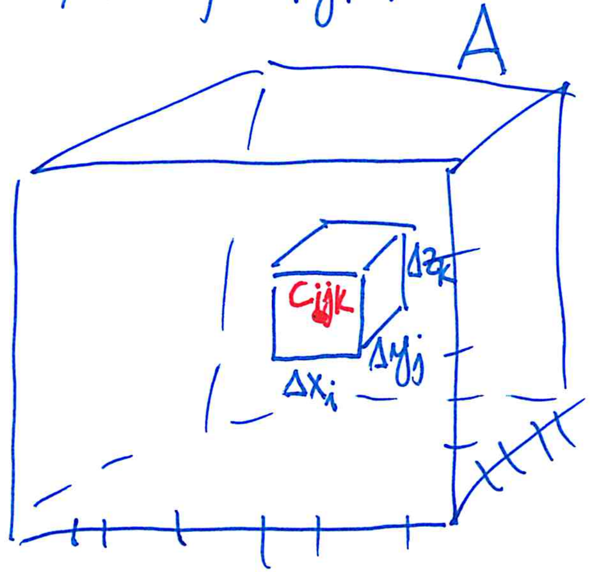
TROJNÝ INTEGRÁL

integrál funkce tří proměnných $u = f(x,y,z)$ přes těleso T.



Konstrukce přes kvádr:

$$\sum_i \sum_j \sum_k \overbrace{f(c_{ijk})}^{1} \Delta x_i \Delta y_j \Delta z_k \longrightarrow \iiint_A f(x,y,z)\,dx\,dy\,dz$$

$f(x,y,z)=1$

$$\iiint_A 1\,dx\,dy\,dz = V(A)$$

Věta o trojnásobném integrálu (Fubiniova):

Předpokládejme, že f je spojitá na $B \subset E_3$. Je-li B obor popsaný

$$a \le x \le b$$
$$d(x) \le y \le h(x)$$
$$D(x,y) \le z \le H(x,y),$$

platí

$$\iiint_B f(x,y,z)\,dx\,dy\,dz = \int_a^b \left[\int_{d(x)}^{h(x)} \left(\int_{D(x,y)}^{H(x,y)} f(x,y,z)\,dz \right) dy \right] dx$$

a analogicky pro další varianty popisu oboru B

Příklady: $\iiint\limits_C x^2 y z^3 \, dx\,dy\,dz$, kde C je kvádr $\langle 0,1\rangle \times \langle 0,2\rangle \times \langle 1,3\rangle$

$0 \le x \le 1$

$0 \le y \le 2$

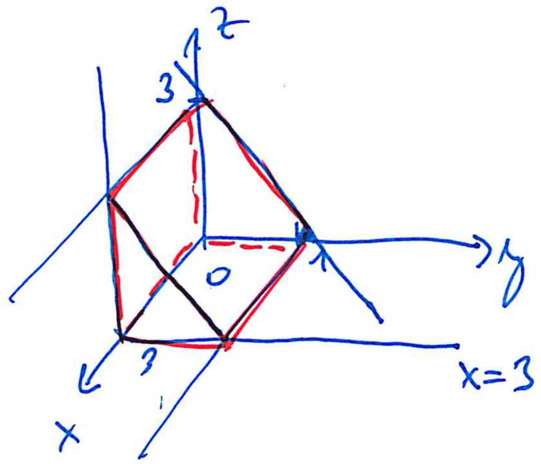
$1 \le z \le 3$

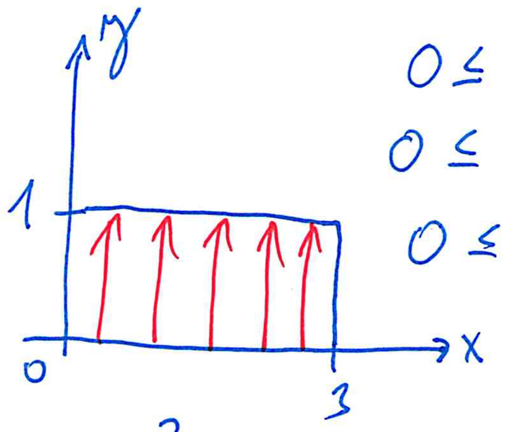
$$\iiint\limits_C x^2 y z^3 \, dx\,dy\,dz = \int_0^1 \left\{ \int_0^2 \left[\int_1^3 x^2 y z^3 \, dz \right] dy \right\} dx = \int_0^1 40x^2 \, dx = 40\left[\frac{x^3}{3}\right]_0^1 = \frac{40}{3}$$

$$\int_1^3 x^2 y z^3 \, dz = x^2 y \left[\frac{z^4}{4}\right]_1^3 = x^2 y \left(\frac{81}{4} - \frac{1}{4}\right) = 20x^2 y$$

$$\int_0^2 20x^2 y \, dy = 10x^2 \left[y^2\right]_0^2 = 40x^2$$

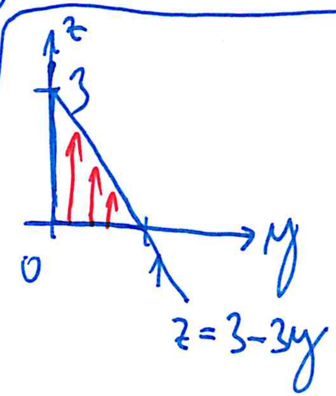
$$\iiint_D 2y\,dx\,dy\,dz \qquad D:\ x=0,\ y=0,\ z=0,\ x=3,\ z=3-3y$$





$$0 \le x \le 3$$
$$0 \le y \le 1$$
$$0 \le z \le 3-3y$$

$$\iiint_D 2y\,dx\,dy\,dz = \int_0^3 \left[\int_0^1 \left(\int_0^{3-3y} 2y\,dz \right) dy \right] dx = \int_0^3 1\,dx = [x]_0^3 = \underline{3}$$



$$0 \le y \le 1$$
$$0 \le z \le 3-3y$$
$$0 \le x \le 3$$

$$\int_0^{3-3y} 2y\,dz = 2y\,[z]_0^{3-3y} = 2y(3-3y) = 6y - 6y^2$$

$$\int_0^1 (6y - 6y^2)\,dy = \left[3y^2 - 2y^3\right]_0^1 = 3 - 2 = \underline{\underline{1}}$$