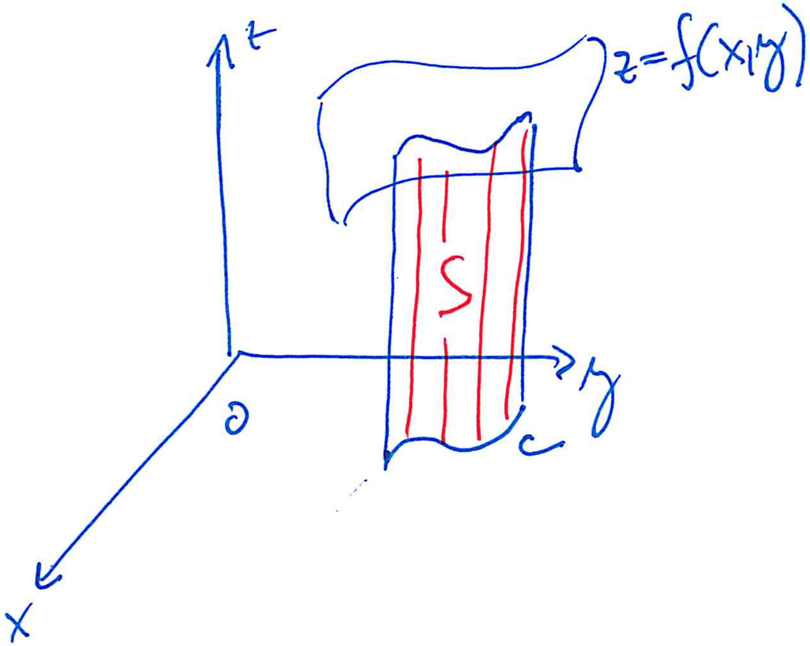
KŘIVKOVÝ INTEGRÁL 1. druhu

Geom. interpretace pro $z = f(x,y)$



Obsah plochy mezi křivkou c v rovině xy a grafem nezáporné funkce $z = f(x,y)$

Nechť je dána reálná funkce f na po částech hladké křivce $c: x = x(t), y = y(t)$ (resp. $d: x = x(t), y = y(t)$ a $z = z(t)$) pro $t \in \langle a, b \rangle$. Křivkový integrál 1. druhu funkce f podél křivky c definujeme vztahem

$$\int_c f(x,y)\,\underline{ds} = \int_a^b f(x(t), y(t))\,\underline{\sqrt{x'(t)^2 + y'(t)^2}\,dt}$$

resp.

$$\int_d f(x,y,z)\,\underline{ds} = \int_a^b f(x(t), y(t), z(t)) \cdot \underline{\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt}$$

$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds,$$ kde $C = C_1 \cup C_2$ a křivky C_1 a C_2 jsou na sebe navazující křivky, které nemají společné vnitřní body.

$$\int_C \alpha f\,ds = \alpha \int_C f\,ds \qquad \alpha \in \mathbb{R}$$

$$\int_C (f+g)\,ds = \cancel{\int_C (f+g)\,ds} = \int_C f\,ds + \int_C g\,ds$$

Příklad: $\int_a x^2 ds$, kde a je část $y = \ln x$ pro $x \in \langle 1,3 \rangle$

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1) parametrisace: $x = t \quad x' = 1$

$y = \ln t \quad y' = \frac{1}{t}$

$t \in \langle 1,3 \rangle$

2) $\boxed{ds = \sqrt{x'^2 + y'^2}\, dt} = \sqrt{1 + \frac{1}{t^2}}\, dt = \sqrt{\frac{t^2+1}{t^2}}\, dt = \frac{\sqrt{t^2+1}}{t}\, dt$

3) $\int_a x^2 ds = \int_1^3 t^{\cancel{2}} \cdot \frac{\sqrt{t^2+1}}{\cancel{t}}\, dt = \int_1^3 t\sqrt{t^2+1}\, dt = \frac{1}{2}\int_2^{10} \sqrt{u}\, du =$

$u = t^2+1$

$du = 2t\,dt$

$\frac{1}{2}du = t\,dt$

$= \frac{1}{2}\frac{2}{3}\left[u^{\frac{3}{2}}\right]_2^{10} = \underline{\underline{\frac{1}{3}\left(\sqrt{1000} - 2\sqrt{2}\right)}}$

Využití integrálu 1. druhu: křivka c s délkovou hustotou $h = h(x,y)$
příp. $h = h(x,y,z)$

délka křivky: $l = \int_c 1\, ds$

hmotnost: $m = \int_c h\, ds$

těžiště $T = [x_T, y_T]$ nebo $T = [x_T, y_T, z_T]$

$$x_T = \frac{1}{m} \int_c x \cdot h\, ds$$

$$y_T = \frac{1}{m} \int_c y \cdot h\, ds$$

resp. $z_T = \frac{1}{m} \int_c z \cdot h\, ds$

Momenty setrvačnosti vzhledem k osám v prostoru:

$$I_x = \int_c (y^2 + z^2)\, h\, ds$$

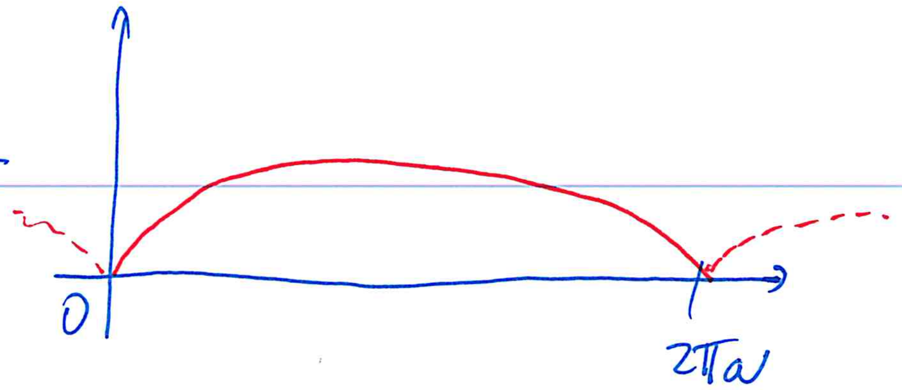
$$I_y = \int_c (x^2 + z^2)\, h\, ds$$

$$I_z = \int_c (x^2 + y^2)\, h\, ds$$

Příklad: Spočítejte délku cykloidy

$c: \quad x = a(t - \sin t) \qquad t \in \langle 0, 2\pi \rangle$

$\qquad y = a(1 - \cos t) \qquad a > 0$



$$l = \int_c 1\,ds = a\int_0^{2\pi} \sqrt{2-2\cos t}\,dt = a\int_0^{2\pi} \sqrt{2-2\left(1-2\sin^2\frac{t}{2}\right)}\,dt = a\int_0^{2\pi} \sqrt{4\sin^2\frac{t}{2}}\,dt =$$

$$= a\int_0^{2\pi} 2\sin\frac{t}{2}\,dt = 2a\left[-\cos\frac{t}{2}\right]_0^{2\pi}\cdot 2 = 4a(1-(-1)) = \boxed{8a}$$

$u = \frac{t}{2}$

$du = \frac{1}{2}dt$

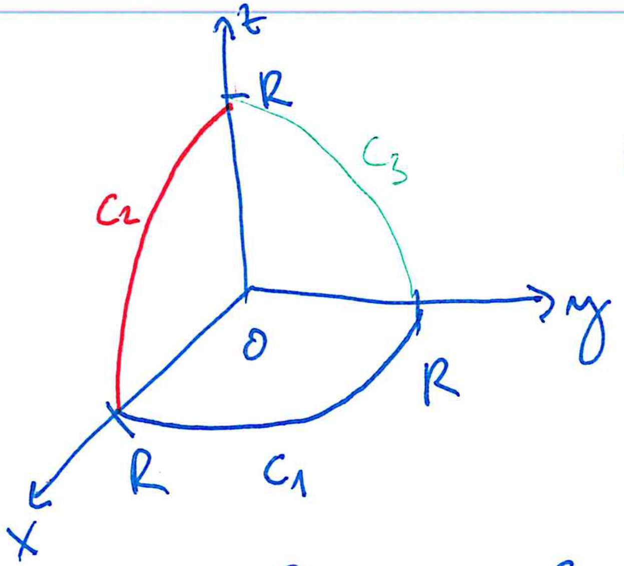
$x' = a(1-\cos t)$

$y' = a\sin t$

$$ds = \sqrt{x'^2 + y'^2}\,dt = \sqrt{a^2(1-2\cos t + \cos^2 t) + a^2\sin^2 t}\,dt = a\sqrt{2-2\cos t}\,dt$$

$$\cos t = \cos^2\frac{t}{2} - \sin^2\frac{t}{2} = 1 - 2\sin^2\frac{t}{2}$$

Nalezněte souřadnice těžiště křivky složené ze tří čtvrtkružnic dle obrázku ležících v souřadných rovinách. $h(x,y,z) = 1$



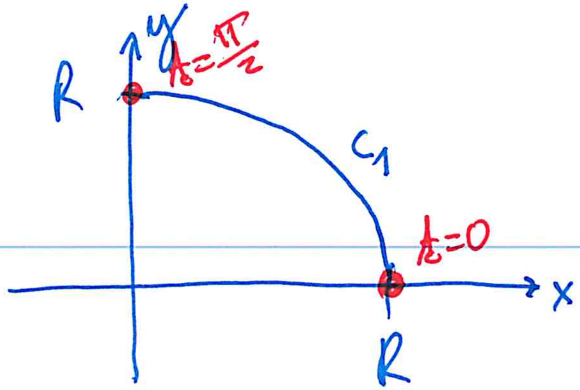
$$m = h \cdot l = \underline{1 \cdot \tfrac{3}{2}\pi R}$$

$$x_T = y_T = z_T \quad \text{(díky symetrii)}$$

$$x_T = \frac{1}{m}\int_C x h \, ds = \frac{2}{3\pi R}\int_C x \, ds = \frac{2}{3\pi R} 2R^2 = \boxed{\frac{4R}{3\pi} = x_T = y_T = z_T}$$

$$\int_C x \, ds = \int_{C_1} x \, ds \checkmark + \int_{C_2} x \, ds \checkmark + \int_{C_3} x \, ds \checkmark = R^2 + R^2 + 0 = 2R^2$$

C_1 :



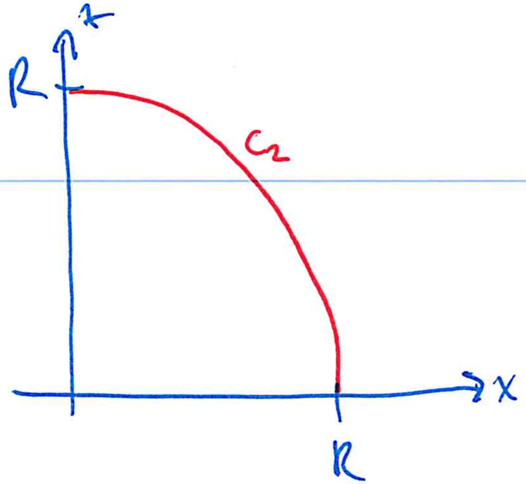
$x = R\cos t \qquad x' = -R\sin t$

$y = R\sin t \qquad y' = R\cos t \qquad t \in \langle 0, \frac{\pi}{2} \rangle$

$z = 0 \qquad z' = 0$

$$ds = \sqrt{x'^2 + y'^2 + z'^2}\, dt = \sqrt{R^2\sin^2 t + R^2\cos^2 t}\, dt = R\,dt$$

$$\int_{C_1} x\, ds = \int_0^{\frac{\pi}{2}} R\cos t\, R\, dt = R^2 \int_0^{\frac{\pi}{2}} \cos t\, dt = R^2 \left[\sin t\right]_0^{\frac{\pi}{2}} = R^2$$



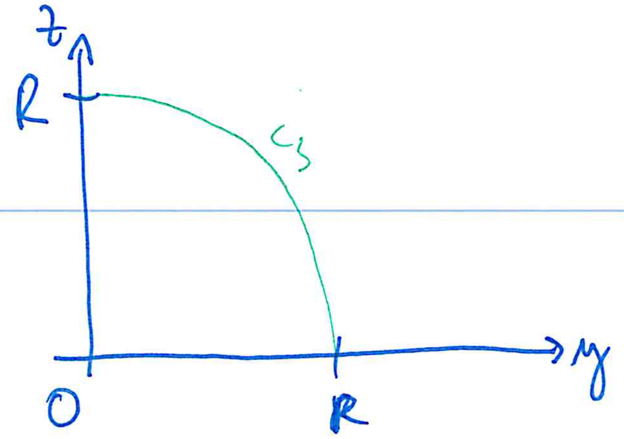
$$x = R\cos t \qquad x' = -R\sin t$$

$$y = 0 \qquad y' = 0$$

$$z = R\sin t \qquad t \in \langle 0, \tfrac{\pi}{2} \rangle \qquad z' = R\cos t$$

$$ds = \sqrt{R^2\sin^2 t + 0^2 + R^2\cos^2 t}\; dt = R\,dt$$

$$\int_{c_2} x\,ds = \int_0^{\frac{\pi}{2}} R\cos t\, R\,dt = \underline{R^2}$$



$x = 0$

$y = R\cos t$ $\quad t \in \langle 0, \frac{\pi}{2} \rangle$

$z = R\sin t$

$ds = R\,dt$

$$\int_{c_3} x\,ds = \int_{c_3} 0\,ds = 0$$