Aritmetické vektory

1) Jsou následující skupiny vektorů lin. závislé nebo nezávislé?

$\alpha \cdot \vec{a} = \vec{o}$

i) $\vec{a} = (1,0,0,1)$ $\qquad \alpha \cdot (1,0,0,1) = (0,0,0,0)$ $\quad \boxed{\alpha = 0}$ $\quad \alpha = 0$

$0 = 0$

$(\alpha, 0, 0, \alpha) = (0,0,0,0)$

Jediné řešení $0 \cdot \vec{a} = \vec{o}$

Vektor $\vec{a}$ je lin. nezávislý.

ii) $\vec{o} = (0,0,0,0)$ $\qquad \beta \cdot (0,0,0,0) = (0,0,0,0)$

β může být cokoliv, např. $8 \cdot (0,0,0,0) = (0,0,0,0)$

Vektor $\vec{o}$ je lin. závislý.

iii) $\vec{a} = (1,0,0,1)$, $\vec{b} = (1,2,-4,3)$

pro $\alpha \neq 0, \beta \neq 0$

$$\alpha \cdot \vec{a} + \beta \vec{b} = \vec{o} \longrightarrow \alpha \vec{a} = -\beta \vec{b} \quad /:\alpha$$

$$\vec{a} = -\frac{\beta}{\alpha} \cdot \vec{b}$$

$$\alpha(1,0,0,1) + \beta(1,2,-4,3) = (0,0,0,0)$$

$$(\alpha,0,0,\alpha) + (\beta, 2\beta, -4\beta, 3\beta) = (0,0,0,0)$$

$$(\alpha+\beta, 2\beta, -4\beta, \alpha+3\beta) = (0,0,0,0)$$

$$\alpha+\beta = 0 \longrightarrow \alpha = 0$$

$$2\beta = 0 \rightarrow \beta = 0$$

$$-4\beta = 0$$

$$\alpha + 3\beta = 0$$

Pouze $0 \cdot \vec{a} + 0 \vec{b} = \vec{o}$, skupina vektorů $\vec{a}, \vec{b}$ je lin. nezávislá.

IV) $\vec{a} = (1,0,0,1)$, $\vec{b} = (1,2,-4,3)$, $\vec{c} = (3,4,-8,7)$

$$\alpha \cdot \vec{a} + \beta \vec{b} + \gamma \vec{c} = \vec{o}$$

$$\alpha(1,0,0,1) + \beta(1,2,-4,3) + \gamma(3,4,-8,7) = (0,0,0,0)$$

$$(\underbrace{\alpha+\beta+3\gamma}_{0}, \underbrace{2\beta+4\gamma}_{0}, \underbrace{-4\beta-8\gamma}_{0}, \underbrace{\alpha+3\beta+7\gamma}_{0}) = (0,0,0,0)$$

$$\begin{aligned} \alpha+\beta+3\gamma &= 0 \\ 2\beta+4\gamma &= 0 \\ -4\beta-8\gamma &= 0 \\ \alpha+3\beta+7\gamma &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 4 \\ 0 & -4 & -8 \\ 1 & 3 & 7 \end{bmatrix} (-1) \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 4 \\ \cancel{0} & \cancel{-4} & \cancel{-8} \\ \cancel{0} & \cancel{2} & \cancel{4} \end{bmatrix}$$

$$\begin{aligned} \alpha+\beta+3\gamma &= 0 \\ \beta+2\gamma &= 0 \end{aligned}$$

$\gamma = t \ (t \in \mathbb{R})$

$\beta = -2t$

$\alpha = 2t - 3t = -t$

např. $t = 1$

$$\boxed{\begin{aligned} \gamma &= 1 \\ \beta &= -2 \\ \alpha &= -1 \end{aligned}}$$

Protože $-(1,0,0,1) - 2(1,2,-4,3) + (3,4,-8,7) = (0,0,0,0)$, vektory $\vec{a}, \vec{b}, \vec{c}$ jsou lin. ZÁVISLÉ.

v) $\vec{a} = (1,0,0,1)$, $\vec{b} = (1,2,-4,3)$, $\vec{d} = (1,-2,-1,0)$

$$\alpha \vec{a} + \beta \vec{b} + \delta \vec{d} = \vec{o}$$

$$\alpha(1,0,0,1) + \beta(1,2,-4,3) + \delta(1,-2,-1,0) = (0,0,0,0)$$

$$(\underbrace{\alpha+\beta+\delta}_{0}, \underbrace{2\beta-2\delta}_{0}, \underbrace{-4\beta-\delta}_{0}, \underbrace{\alpha+3\beta}_{0}) = (0,0,0,0)$$

$$\begin{aligned} \alpha+\beta+\delta &= 0 \\ 2\beta-2\delta &= 0 \\ -4\beta-\delta &= 0 \\ \alpha+3\beta &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -2 \\ 0 & -4 & -1 \\ 1 & 3 & 0 \end{bmatrix} \begin{matrix} (-1) \\ \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -2 \\ 0 & -4 & -1 \\ 0 & 2 & -1 \end{bmatrix} \begin{matrix} \\ 2 \quad (-1) \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -2 \\ \cancel{0} & \cancel{0} & \cancel{-5} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \alpha+\beta+\gamma &= 0 \\ \beta-\gamma &= 0 \end{aligned}$$

$$\boxed{\begin{aligned} \gamma &= 0 \\ \beta &= 0 \\ \alpha &= 0 \end{aligned}}$$

Protože pouze $0\vec{a} + 0\vec{b} + 0\vec{d} = \vec{o}$, skupina vektorů $\vec{a}, \vec{b}, \vec{d}$ je lin. nezávislá.

vi) $\vec{a} = (1,0,0,1)$, $\vec{b} = (1,2,-4,3)$, $\vec{o} = (0,0,0,0)$

$$\alpha \vec{a} + \beta \vec{b} + \gamma \vec{o} = \vec{o}$$

$$0\vec{a} + 0\vec{b} + 215\,\vec{o} = \vec{o}$$

Skupina $\vec{a}, \vec{b}, \vec{o}$ je lin. závislá.

vii) $\vec{a} = (1,0,0,1)$, $\vec{b} = (1,2,-4,3)$, $\vec{d} = (1,-2,-1,0)$, $\vec{e} = (2,1,-1,7)$

$$\alpha\vec{a} + \beta\vec{b} + \delta\vec{d} + \varepsilon\vec{e} = \vec{o}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 2 & -2 & 1 \\ 0 & -4 & -1 & -1 \\ 1 & 3 & 0 & 7 \end{bmatrix} (-1) \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 2 & -2 & 1 \\ 0 & -4 & -1 & -1 \\ 0 & 2 & -1 & 5 \end{bmatrix} (2)\ (-1) \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 2 & -2 & 1 \\ 0 & 0 & -5 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix} 5 \sim$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 2 & -2 & 1 \\ 0 & 0 & -5 & 1 \\ 0 & 0 & 0 & 21 \end{bmatrix}$$

$$\begin{aligned} \alpha + \beta + \cancel{\gamma} + \delta + 2\varepsilon &= 0 & \rightarrow \alpha &= 0 \\ 2\beta - 2\delta + \varepsilon &= 0 & \rightarrow \beta &= 0 \\ -5\delta + \varepsilon &= 0 & \rightarrow \delta &= 0 \\ 21\varepsilon &= 0 & \rightarrow \varepsilon &= 0 \end{aligned}$$

Pouze $0\vec{a} + 0\vec{b} + 0\vec{d} + 0\vec{e} = \vec{o}$, skupina vektorů $\vec{a}, \vec{b}, \vec{d}, \vec{e}$ je lin. nezávislá.

viii) $\vec{a}=(1,0,0,1)$, $\vec{b}=(1,2,-4,3)$, $\vec{d}=(1,-2,-1,0)$, $\vec{e}=(2,1,-1,7)$, $f=(9,8,7,4)$

$$\alpha\vec{a}+\beta\vec{b}+\delta\vec{d}+\varepsilon\vec{e}+\varphi\vec{f}=\vec{o}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 9 \\ 0 & 2 & -2 & 1 & 8 \\ 0 & -4 & -1 & -1 & 7 \\ 1 & 3 & 0 & 7 & 4 \end{bmatrix}$$

homogenní soustava 4 rovnice pro 5 neznámých

$\downarrow$

∞ řešení

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existuje nenulové řešení

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vektory jsou lin. závislé

$\downarrow$

v $\mathbb{R}^4$ je lib. skupina 5 a více vektorů lin. závislá.

2) v $\mathbb{R}^3$: vektor $\vec{w} = (5,4,7)$ vyjádřete jako lin. kombinaci

$\vec{x} = (1,2,3)$, $\vec{y} = (1,0,1)$ a $\vec{z} = (0,0,1)$

$$\alpha \vec{x} + \beta \vec{y} + \gamma \vec{z} = \vec{w} \quad \rightarrow \quad \boxed{2\vec{x} + 3\vec{y} - 2\vec{z} = \vec{w}}$$

$$\alpha(1,2,3) + \beta(1,0,1) + \gamma(0,0,1) = (5,4,7)$$

$$(\underbrace{\alpha+\beta}_{5}, \underbrace{2\alpha}_{4}, \underbrace{3\alpha+\beta+\gamma}_{7}) = (\underline{5}, \underline{4}, \underline{7})$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 2 & 0 & 0 & 4 \\ 3 & 1 & 1 & 7 \end{array}\right]$$

$\alpha + \beta = 5 \rightarrow \beta = 3$

$2\alpha = 4 \rightarrow \alpha = 2$

$3\alpha + \beta + \gamma = 7 \rightarrow \gamma = -2$

3) vyjádřete $\vec{p} = (0,1,3)$ jako lin. kombinaci $\vec{m} = (1,1,1)$

$\vec{n} = (-1,0,1)$

$\vec{q} = (0,1,2)$

$$\alpha \vec{m} + \beta \vec{n} + \gamma \vec{q} = \vec{p}$$

$$\alpha(1,1,1) + \beta(-1,0,1) + \gamma(0,1,2) = (0,1,3)$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 3 \end{array}\right] (-1) \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & 3 \end{array}\right] (-2) \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

nemá řešení

$\vec{p}$ nejde vyjádřit jako lin. kombinaci $\vec{m}, \vec{n}, \vec{q}$