Parciální derivace 2. řádu

$f(x,y) = x^2 y + y^2$

$$\frac{\partial f}{\partial x} = 2xy \quad \to \quad \frac{\partial^2 f}{\partial x^2} = 2y$$

$$\frac{\partial f}{\partial x} = 2xy \quad \to \quad \frac{\partial^2 f}{\partial y \, \partial x} = 2x$$

$$\frac{\partial f}{\partial y} = x^2 + 2y \quad \to \quad \frac{\partial^2 f}{\partial x \, \partial y} = 2x$$

$$\frac{\partial f}{\partial y} = x^2 + 2y \quad \to \quad \frac{\partial^2 f}{\partial y^2} = 2$$

$g(x,y,z) = x^2 y^3 z - 2z$

$\frac{\partial g}{\partial x} = y^3 z\, 2x$

$\frac{\partial g}{\partial y} = x^2 z\, 3y^2$

$\frac{\partial g}{\partial z} = x^2 y^3 - 2$

$\downarrow$

$\frac{\partial^2 g}{\partial x^2} = 2y^3 z$

$\frac{\partial^2 g}{\partial x \partial y} = 3y^2 z\, 2x$

$\frac{\partial^2 g}{\partial x \partial z} = 2xy^3$

$\frac{\partial^2 g}{\partial y \partial x} = 2xz \cdot 3y^2$

$\frac{\partial^2 g}{\partial y^2} = 3x^2 z\, 2y$

$\frac{\partial^2 g}{\partial y \partial z} = x^2 3y^2$

$\frac{\partial^2 g}{\partial z \partial x} = 2xy^3$

$\frac{\partial^2 g}{\partial z \partial y} = 3x^2 y^2$

$\frac{\partial^2 g}{\partial z^2} = 0$

Lokální extrémy

Funkce f (n proměnných) má v bodě A (vnitřní bod D_f) lokální maximum,
minimum

jestliže existuje okolí U bodu A takové, že pro $\forall X \in U$ je $f(X) \le f(A)$

$f(X) \ge f(A)$

Má-li f v bodě A lokální extrém, jsou v tomto bodě všechny 1. parciální derivace, které existují, rovny nule.
Body, v nichž všechny 1. parciální derivace existují a jsou rovny nule, se nazývají stacionární body. Funkce může mít lok. extrém pouze ve stac. bodech nebo v bodech, kde některé parciální derivace neexistují.

Pro stacionární bod A: spočítáme všechny 2. derivace a určíme čísla:

$$D_1(A) = \frac{\partial^2 f(A)}{\partial x^2} \qquad D_2(A) = \begin{array}{c c} x & y \\ \left|\begin{array}{cc} \dfrac{\partial^2 f(A)}{\partial x^2} & \dfrac{\partial^2 f(A)}{\partial x \partial y} \\[2ex] \dfrac{\partial^2 f(A)}{\partial y \partial x} & \dfrac{\partial^2 f(A)}{\partial y^2} \end{array}\right| \end{array} \begin{array}{c} x \\[4ex] y \end{array}$$

$$D_3(A) = \begin{array}{c} \\ x \\[4ex] y \\[4ex] z \end{array} \begin{array}{ccc} x & y & z \\ \left|\begin{array}{ccc} \dfrac{\partial^2 f(A)}{\partial x^2} & \dfrac{\partial^2 f(A)}{\partial x \partial y} & \dfrac{\partial^2 f(A)}{\partial x \partial z} \\[2ex] \dfrac{\partial^2 f(A)}{\partial y \partial x} & \dfrac{\partial^2 f(A)}{\partial y^2} & \dfrac{\partial^2 f(A)}{\partial y \partial z} \\[2ex] \dfrac{\partial^2 f(A)}{\partial z \partial x} & \dfrac{\partial^2 f(A)}{\partial z \partial y} & \dfrac{\partial^2 f(A)}{\partial^2 z^2} \end{array}\right| \end{array}$$

pro funkci 2 proměnných

pro funkci 3 proměnných

1) $D_1(A) > 0$, $D_2(A) > 0$ a příp. $D_3(A) > 0$ — f má v bodě A lok. minimum
2) $D_1(A) < 0$, $D_2(A) > 0$, a příp. $D_3(A) < 0$ — f má v bodě A lok. maximum
3) Jsou-li znaménka jinak než v 1) nebo 2) a D_1 nebo D_2 nebo D_3 není nula — extrém není
4) Je-li $D_1(A) = 0$ nebo $D_2(A) = 0$ nebo $D_3(A) = 0$ $\rightarrow$ NEVÍME
5) Body z 4) a body v nichž neexistují derivace vyšetřujeme dalšími metody

$$f(x,y) = x^2 + y^2$$

$$\frac{\partial f}{\partial x} = 2x = 0$$

$$A = [0,0]$$

$$\frac{\partial f}{\partial y} = 2y = 0$$

$$\frac{\partial^2 f}{\partial x^2} = 2 \qquad \frac{\partial^2 f}{\partial x \partial y} = 0 \qquad \frac{\partial^2 f}{\partial y^2} = 2$$

$$D_1(A) = 2 \qquad D_2(A) = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$$

Protože $D_1(A) > 0$ a $D_2(A) > 0$

f má v $A = [0,0]$ lok. minimum

$$g(x,y) = x^2 - y^2$$

$$\frac{\partial g}{\partial x} = 2x = 0$$

$$B = [0,0]$$

$$\frac{\partial g}{\partial y} = -2y = 0$$

$$\frac{\partial^2 g}{\partial x^2} = 2 \qquad \frac{\partial^2 g}{\partial x \partial y} = 0 \qquad \frac{\partial^2 g}{\partial y^2} = -2$$

$$D_1(B) = 2 \qquad D_2(B) = \begin{vmatrix} \frac{\partial^2 g}{\partial x^2} & \frac{\partial^2 g}{\partial x \partial y} \\ \frac{\partial^2 g}{\partial y \partial x} & \frac{\partial^2 g}{\partial y^2} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = \underline{\underline{-4}}$$

Protože $D_1(B) > 0$ a $D_2(B) < 0$

$\Rightarrow$ g v bodě B lok. extrém nemá

$$f(x,y) = x^3 + y^3 - 3xy$$

$$\frac{\partial f}{\partial x} = 3x^2 - 3y = 0 \rightarrow 3 \cdot y^4 - 3y = 0 \rightarrow y(y^3 - 1) = 0$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3x = 0 \rightarrow x = y^2$$

$$y_1 = 0 \qquad y^3 - 1 = 0$$

$$y^3 = 1$$

$$y_2 = 1$$

$$P_1 = [0,0]$$

$$P_2 = [1,1]$$

$$x_1 = y_1^2 = 0^2 = 0$$

$$x_2 = y_2^2 = 1^2 = 1$$

$$\frac{\partial^2 f}{\partial x^2} = 6x \quad \frac{\partial^2 f}{\partial x \partial y} = -3 \quad \frac{\partial^2 f}{\partial y^2} = 6y$$

$$D_1 = 6x$$

$$D_2 = \begin{vmatrix} 6x & -3 \\ -3 & 6y \end{vmatrix} = 36xy - 9$$

	D_1	D_2	extrém
P_1	0	-9	NEVÍME
P_2	$+6$	$+27$	lok. minimum

$$f(x,y,z) = 2xy^2 - 4xy + x^2 + z^2 - 2z$$

$$\frac{\partial f}{\partial x} = 2y^2 - 4y + 2x = 0$$

$$\frac{\partial f}{\partial y} = 4xy - 4x = 0 = 4x(y-1) \quad \begin{cases} x_1 = 0 \\ y_2 = 1 \end{cases}$$

$$\frac{\partial f}{\partial z} = 2z - 2 = 0 \rightarrow \underline{z = 1}$$

$x_1 = 0$

a) $2y^2 - 4y = 0$

$2y(y-2) = 0$

$y = 0$

$y = 2$

$P_1 = [0,0,1]$

$P_2 = [0,2,1]$

$y_2 = 1$

b) $2 - 4 + 2x = 0$

$2x - 2 = 0$

$x = 1$

$P_3 = [1,1,1]$

$$\frac{\partial^2 f}{\partial x^2} = 2 \qquad \frac{\partial^2 f}{\partial x \partial y} = 4y - 4 \qquad \frac{\partial^2 f}{\partial x \partial z} = 0$$

$$\frac{\partial^2 f}{\partial y^2} = 4x \qquad \frac{\partial^2 f}{\partial y \partial z} = 0$$

$$\frac{\partial f}{\partial z} = 2$$

$$D_1 = 2$$

$$D_2 = \begin{vmatrix} 2 & 4y-4 \\ 4y-4 & 4x \end{vmatrix} = 8x - (4y-4)^2$$

$$D_3 = \begin{vmatrix} 2 & 4y-4 & 0 \\ 4y-4 & 4x & 0 \\ 0 & 0 & 2 \end{vmatrix} = 2\left(8x - (4y-4)\right)^2$$

	D_1	D_2	D_3	extrém
P_1	+	−	−	NENÍ
P_2	+	−	−	NENÍ
P_3	+	+	+	lok. minimum