OLDR n-tého řádu s konst. koef. – nehomogenní

$$y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = f(x). \quad (*)$$

Všeobecné řešení rovnice $(*)$ získáme jako $y = y_H + y_P$, kde

y_H jsou všeobecné řešení přidružené homogenní rovnice a

y_P je tzv. partikulární, které sestavíme dle vzorce a speciálního

tvaru funkce $f(x)$ na pravé straně.

Je-li $f(x) = e^{\alpha x}\left(P_n(x)\cos\beta x + Q_n(x)\cdot\sin\beta x\right)$,

hledáme $y_P = e^{\alpha x}\cdot x^k\left(R_n(x)\cos\beta x + S_n(x)\sin\beta x\right)$, kde

α, β vidíme ze zadání

k je násobnost $\alpha + i\beta$ jako kořene char. rovnice

$R_n(x)$ a $S_n(x)$ jsou polynomy s neurčitými koeficienty stejného stupně n

Příklady: $y'' - 2y' + y = e^x(1 \cdot \cos 0x + 0 \sin 0x)$

a) $y'' - 2y' + y = 0$

$\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)^2 = 0$

$\lambda_{1,2} = 1$

kolikrát je 1 kořenem ch. r?

$y_1 = e^x, \; y_2 = xe^x$

$y_H = c_1 e^x + c_2 x e^x$

$c_1, c_2 \in \mathbb{R}$

b) $\alpha = 1$ $\quad R_n(x) = A$

$\beta = 0$ $\quad S_n(x) = B$

k je násobnost $\alpha + i\beta$ jako kořene ch. r. $\alpha + i\beta = 1 + 0i = 1$

$\downarrow$

$K = 2$

$y_p = e^x \cdot x^2 (A \underbrace{\cos 0x}_{1} + B \underbrace{\sin 0x}_{0}) = Ax^2 e^x$

$y_p' = A2xe^x + Ax^2e^x = Ae^x(x^2 + 2x)$

$y_p'' = Ae^x(x^2 + 2x) + Ae^x(2x + 2)$

dosadíme do rovnice

dosazení: $Ae^x(x^2 + 2x) + Ae^x(2x + 2) - 2Ae^x(x^2 + 2x) + Ax^2e^x = e^x$

$2Ae^x = e^x \longrightarrow 2A = 1 \longrightarrow \boxed{A = \frac{1}{2}} \to y_p = \frac{1}{2}x^2e^x$

obecné: $y = y_H + y_p = c_1 e^x + c_2 e^x \cdot x + \frac{1}{2}x^2e^x, \; c_1, c_2 \in \mathbb{R}$

$$y'' + y' = 3;\quad y(0)=1,\ y'(0)=0 \qquad 3 = 3e^{0x}(\cos 0x + \sin 0x)$$

a) $y'' + y' = 0$

$\lambda^2 + \lambda = 0$

$\lambda(\lambda+1) = 0$

$\lambda_1 = 0,\ \lambda_2 = -1$

$y_H = c_1 e^{0x} + c_2 e^{-x} = c_1 + c_2 e^{-x}$

b) $\left.\begin{array}{l}\alpha = 0\\ \beta = 0\end{array}\right\}\ \alpha + i\beta = 0 + 0i = 0 \rightarrow k = 1 \qquad R_n(x) = A,\ S_n(x) = B$

$1x$

$$y_p = \underbrace{e^{0x}}_{1} \cdot x^1 \left(A \underbrace{\cos 0x}_{1} + B \underbrace{\sin 0x}_{0}\right) = Ax = \underline{\underline{3x}}$$

$y_p' = A,\ y_p'' = 0$

dosazení do rovnice: $0 + A = 3 \rightarrow \underline{\underline{A = 3}}$

c) $y = c_1 + c_2 e^{-x} + 3x \longrightarrow 1 = c_1 + c_2$

$y' = -c_2 e^{-x} + 3 \longrightarrow 0 = -c_2 + 3 \rightarrow \underline{\underline{c_2 = 3}}$

$\underline{\underline{c_1 = -2}}$

$$\longrightarrow \boxed{y = -2 + 3e^{-x} + 3x}$$

$$y'' - 4y' + 4y = 8x^2 \left(e^{0x}\left(\cos 0x + \sin 0x\right)\right)$$

a) $y'' - 4y' + 4y = 0$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$(\lambda - 2)^2 = 0$$

$$\lambda_{1,2} = 2$$

$$y_H = c_1 e^{2x} + c_2 x e^{2x}$$

b) $\alpha = 0$, $\beta = 0$ $\quad \alpha + i\beta = 0 \to k = 0$

$R_n(x) = Ax^2 + Bx + C$

$S_n(x) = Dx^2 + Ex + F$

$0x$

$$y_p = \underset{1}{e^{0x}} \cdot \underset{1}{x^0} \left[(Ax^2 + Bx + C)\underset{1}{\cos 0x} + (Dx^2 + Ex + F)\underset{0}{\sin 0x}\right]$$

$$y_p = Ax^2 + Bx + C, \quad y_p' = 2Ax + B, \quad y_p'' = 2A$$

dosazení: $\underbrace{2A}_{y_p''} \underbrace{- 8Ax - 4B}_{-4y_p'} \underbrace{+ 4Ax^2 + 4Bx + 4C}_{+4y_p} = 8x^2$

u x^2: $4A = 8 \Rightarrow \underline{A = 2}$

u x: $-8A + 4B = 0 \to B = 2A = \underline{4}$

u x^0: $2A - 4B + 4C = 0 \to 4C = 4B - 2A = 16 - 4 = 12$

$\underline{C = 3}$

$\to y_p = 2x^2 + 4x + 3$

$c_1, c_2 \in \mathbb{R}$

$$\boxed{y = y_H + y_p = c_1 e^{2x} + c_2 x e^{2x} + 2x^2 + 4x + 3}$$

$$y'' + 4y = \sin 2x$$

a) $y'' + 4y = 0$

$$\lambda^2 + 4 = 0$$

$$\lambda^2 = -4$$

$$\lambda_{1,2} = \pm 2i$$

$$y_H = C_1 \cos 2x + C_2 \sin 2x$$

$\alpha = 0$

$\beta = 2$

$$\alpha + i\beta = 0 + 2i = 2i \rightarrow k = 1$$

1x

$\cancel{A}$ $R_n(x) = A$

$S_n(x) = B$

$$y_p = x(A\cos 2x + B\sin 2x)$$

$$y_p' = (A\cos 2x + B\sin 2x) + x(-2A\sin 2x + 2B\cos 2x)$$

$$y_p'' = -2A\sin 2x + 2B\cos 2x + (-2A\sin 2x + 2B\cos 2x) +$$

$$+ x(-4A\cos 2x - 4B\sin 2x)$$

dosazení: $-2A\sin 2x + 2B\cos 2x - 2A\sin 2x + 2B\cos 2x - \cancel{4Ax\cos 2x} - \cancel{4Bx\sin 2x} +$

$$+ \cancel{4Ax\cos 2x} + \cancel{4Bx\sin 2x} = \sin 2x$$

$$-4A\sin 2x + 4B\cos 2x = \sin 2x$$

$$\sin: -4A = 1 \rightarrow A = -\frac{1}{4} \qquad \cos: 4B = 0 \rightarrow B = 0 \rightarrow y_p = -\frac{1}{4}x\cos 2x$$

$$\boxed{y = y_H + y_p = C_1\cos 2x + C_2\sin 2x - \frac{1}{4}x\cos 2x}$$

$$y'''' - 8y''' + 16y'' = 192x$$

a) $y'''' - 8y''' + 16y'' = 0$

$$\lambda^4 - 8\lambda^3 + 16\lambda^2 = 0$$

$$\lambda^2(\lambda^2 - 8\lambda + 16) = 0$$

$$\lambda^2(\lambda - 4)^2 = 0$$

$\lambda_{1,2} = 0 \quad \lambda_{3,4} = 4$

$y_1 = e^{0x} = 1 \qquad y_3 = e^{4x}$

$y_2 = x \cdot e^{0x} = x \qquad y_4 = xe^{4x}$

$$y_H = c_1 + c_2 x + c_3 e^{4x} + c_4 x e^{4x}$$

$c_1, c_2, c_3, c_4 \in \mathbb{R}$

b) $\left.\begin{array}{l}\alpha = 0\\ \beta = 0\end{array}\right\} \alpha + i\beta = 0 + 0i = 0 \quad k = 2 \qquad R_n(x) = Ax + B, \quad S_n(x) = Cx + D$

2×

$$y_p = \underbrace{e^{0x}}_{1} \cdot x^2 \left[(Ax+B)\underbrace{\cos 0x}_{1} + (Cx+D)\underbrace{\sin 0x}_{0}\right]$$

$y_p = Ax^3 + Bx^2, \quad y_p' = 3Ax^2 + 2Bx, \quad y_p'' = 6Ax + 2B$

$y_p''' = 6A, \quad y_p'''' = 0$

dosazení: $0 - 48A + 96Ax + 32B = 192x$

u x: $96A = 192 \rightarrow A = 2$

u x^0: $-48A + 32B = 0 \rightarrow 32B = 48A = 96$

$B = 3$

$$y_p = 2x^3 + 3x^2$$

$$\boxed{y = y_H + y_p}$$