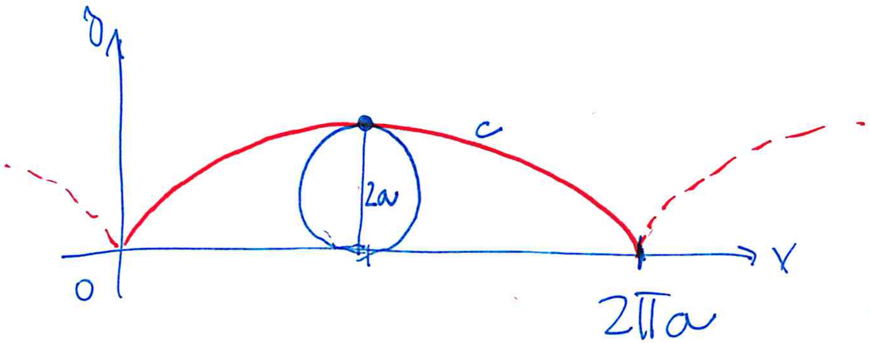
těžiště $\quad x = a(t - \sin t) \quad a > 0, \; t \in \langle 0, 2\pi \rangle$

$\qquad\qquad y = a(1 - \cos t) \quad h(x,y) = 1$

$l = 8a \qquad\qquad x_T = \pi a$

$m = h \cdot l = 8a$



$$y_T = \frac{1}{m} \int_c y \, ds$$

$$ds = \sqrt{(x')^2 + (y')^2}\, dt = \sqrt{a^2(1 - 2\cos t + \cos^2 t) + a^2 \sin^2 t}\, dt =$$

$x' = a(1 - \cos t)$

$$= a\sqrt{2 - 2\cos t}\, dt = \underline{\sqrt{2}\, a \sqrt{1 - \cos t}\, dt}$$

$y' = a \sin t$

$$y_T = \frac{1}{8a}\int_C y\,ds = \frac{\cancel{4}\sqrt{2}\,\cancel{a}}{8\cancel{a}}\int_0^{2\pi} \underbrace{\cancel{a}(1-\cos t)}_{y}\,\underbrace{\sqrt{1-\cos t}\,dt}_{ds} =$$

$$= \frac{\sqrt{2}\,a}{8}\int_0^{2\pi}(1-\cos t)^{\frac{3}{2}}\,dt = \frac{\sqrt{2}\,a}{8}\int_0^{2\pi}\left(2\sin^{2}\frac{t}{2}\right)^{\frac{3}{2}}dt = *$$

$$\cos 2t = \cos^2 t - \sin^2 t$$

$$\cos t = \cos^2\frac{t}{2} - \sin^2\frac{t}{2} = 1 - 2\sin^2\frac{t}{2} \rightarrow \underbrace{2\sin^2\frac{t}{2} = 1-\cos t}$$

$$* \; \frac{\sqrt{2}\,a\sqrt{8}}{8}\int_0^{2\pi}\sin^3\frac{t}{2}\,dt = \frac{\sqrt{16}\,a}{8}\int_0^{2\pi}\sin^2\frac{t}{2}\cdot\sin\frac{t}{2}\,dt = \frac{a}{2}\int_0^{2\pi}\left(1-\cos^2\frac{t}{2}\right)\sin\frac{t}{2}\,dt =$$

$$u = \cos\frac{t}{2}$$

$$du = -\sin\frac{t}{2}\cdot\frac{1}{2}\,dt$$

$$-2\,du = \sin\frac{t}{2}\,dt$$

$$= \frac{-2a}{2}\int_1^{-1}(1-u^2)\,du = a\left[u - \frac{u^3}{3}\right]_{-1}^{1} = \boxed{\frac{4}{3}a = y_T}$$

1) $y' - 2y = x^2 + 2$

a) $y' - 2y = 0 \quad (y' = 2y)$

$$\frac{dy}{dx} = 2y \quad /:y \quad /\cdot dx$$

$$\int \frac{dy}{y} = \int 2\,dx$$

$$\ln|y| = 2x + c_1 \quad c_1 \in \mathbb{R}$$

$$|y| = e^{2x+c_1} = e^{2x} \cdot e^{c_1} = e^{2x} \cdot c_2 \quad c_2 > 0$$

$$y_H = ce^{2x} \quad c \in \mathbb{R}$$

b) $y' - 2y = x^2 + 2$

$y_p = ce^{2x}$ c je teď funkce $c = c(x)$

$$y_p' = c'e^{2x} + ce^{2x} \cdot 2$$

dosazení:

$$\underbrace{c'e^{2x} + \cancel{ce^{2x} \cdot 2}}_{y'} \underbrace{- \cancel{2ce^{2x}}}_{-2y} = x^2 + 2$$

$$c' \cdot e^{2x} = x^2 + 2$$

$$c' = \frac{x^2 + 2}{e^{2x}} = e^{-2x}(x^2 + 2)$$

$$c = \int e^{-2x}(x^2 + 2)\,dx$$

$$c' = e^{-2x}(x^2+2) \rightarrow c = \int e^{-2x}(x^2+2)\,dx = -\frac{1}{2}(x^2+2)e^{-2x} + \int x e^{-2x}\,dx =$$

$$u = x^2+2 \quad u' = 2x \qquad u = x \quad u' = 1$$

$$v' = e^{-2x} \quad v = -\frac{1}{2}e^{-2x} \qquad v' = e^{-2x} \quad v = -\frac{1}{2}e^{-2x}$$

$$= -\frac{1}{2}(x^2+2)e^{-2x} - \frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}\,dx = \underline{-\frac{1}{2}e^{-2x}}(x^2+x+2)\underline{-\frac{1}{2}e^{-2x}}\cdot\frac{1}{2} =$$

$$= -\frac{1}{2}e^{-2x}\left(x^2+x+\frac{5}{2}\right) = c \rightarrow y_p = -\frac{1}{2}e^{-2x}\left(x^2+x+\frac{5}{2}\right)e^{2x} = \underline{-\frac{1}{2}\left(x^2+x+\frac{5}{2}\right)}$$

$$y = y_H + y_p = \underbrace{c\cdot e^{2x}}_{y_H} \underbrace{- \frac{1}{2}\left(x^2+x+\frac{5}{2}\right)}_{y_p}, \; c \in \mathbb{R}$$

$$\underline{y' + 5x^4 y = x^4}\,,\quad y(0) = 3$$

a) $y' + 5x^4 y = 0$

$$\frac{dy}{dx} = -5x^4 y \quad /:y \quad /\cdot dx$$

$$\int \frac{dy}{y} = \int -5x^4\,dx$$

$$\ln|y| = -x^5 + c_1\,,\quad c_1 \in \mathbb{R}$$

$$|y| = e^{-x^5 + c_1} = e^{-x^5} \cdot e^{c_1} = c_2 e^{-x^5}\,,\quad c_2 > 0$$

$$\underline{y_H = c \cdot e^{-x^5}\,,\quad c \in \mathbb{R}}$$

b) $y' + 5x^4 y = x^4$

$y_p = c \cdot e^{-x^5}$ c jiteď funkce

$$y_p' = c' e^{-x^5} - c e^{-x^5} 5x^4$$

dosazení:

$$\underbrace{c' e^{-x^5} - \cancel{c e^{-x^5} 5x^4}}_{y_p'} + \cancel{5 \underbrace{c e^{-x^5}}_{y_p} x^4} = x^4$$

$$c' e^{-x^5} = x^4$$

$$\underline{c' = e^{x^5} \cdot x^4}$$

$$\boxed{\begin{array}{ll} y = e^x \to x = \ln y & e^{a+b} = e^a \cdot e^b \\ \ln(e^x) = x\,,\ x \in \mathbb{R} & e^{\ln x} = x\,,\ x > 0 \end{array}}$$

$$C = \int e^{x^5} \cdot x^4\,dx = \frac{1}{5}\int e^t\,dt = \frac{1}{5}e^t = \frac{1}{5}e^{x^5}$$

$$t = x^5$$
$$dt = 5x^4\,dx$$
$$\frac{1}{5}\,dt = x^4\,dx$$

$$y_p = Ce^{-x^5} = \frac{1}{5}e^{x^5}e^{-x^5} = \frac{1}{5}$$

obecné: $y = y_H + y_p = Ce^{-x^5} + \frac{1}{5}, \quad C \in \mathbb{R}$

c) $y(0) = 3$: dosazení

$$3 = Ce^0 + \frac{1}{5} = C + \frac{1}{5} \rightarrow C = 3 - \frac{1}{5} = \frac{14}{5}$$

$$\boxed{y = \frac{14}{5}e^{-x^5} + \frac{1}{5}}$$

2) $y' + y = \sin x$

a) $y' + y = 0$

$$\frac{dy}{dx} = -y \quad / :y \quad / \cdot dx$$

$$\int \frac{dy}{y} = \int -dx$$

$$\ln|y| = -x + c_1, \quad c_1 \in \mathbb{R}$$

$$|y| = e^{-x+c_1} = e^{-x} \cdot e^{c_1} = c_2 e^{-x}, \quad c_2 > 0$$

$$y_H = c e^{-x}, \quad c \in \mathbb{R}$$

b) $y' + y = \sin x$

$y_p = c e^{-x}$, c je teď funkce

$$y_p' = c' e^{-x} + c e^{-x}(-1)$$

dosazení: $c' e^{-x} - \cancel{c e^{-x}} + \cancel{c e^{-x}} = \sin x$

$$c' e^{-x} = \sin x \quad / : e^{-x} \sim \cdot e^{x}$$

$$\underline{c' = e^{x} \sin x}$$

$$c = \int e^{x} \sin x \, dx$$

$$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx = e^x \sin x - \left(e^x \cos x + \int \sin x\, e^x dx\right)$$

$u = \sin x \rightarrow u' = \cos x$ $\qquad$ $u = \cos x \rightarrow u' = -\sin x$

$v' = e^x \rightarrow v = e^x$ $\qquad$ $v' = e^x \quad v = e^x$

$$2\int e^x \sin x\,dx = e^x(\sin x - \cos x) \quad /:2$$

$$\underline{c = \int e^x \sin x\,dx = \frac{e^x}{2}(\sin x - \cos x)} \rightarrow y_p = \frac{\cancel{e^x}}{2}(\sin x - \cos x)\cancel{e^{-x}} =$$

$$\underline{= \frac{1}{2}(\sin x - \cos x)}$$

obecné řešení: $y = y_H + y_p = \underline{ce^{-x} + \frac{1}{2}(\sin x - \cos x)}, \; c \in \mathbb{R}$