PLOŠNÉ INTEGRÁLY

Plocha: graf funkce $z = f(x,y)$ na uzavřené množině D

(příp. $x = f(y,z)$ nebo $y = f(x,z)$)

tečná rovina ke grafu $z = f(x,y)$ v bodě $A = [x_0, y_0]$:

$$z = f(A) + \frac{\partial f(A)}{\partial x}(x - x_0) + \frac{\partial f(A)}{\partial y}(y - y_0)$$

rovina $ax + by + cz + d = 0$ má normálový vektor

$$\vec{n} = (a, b, c)$$

tečná rovina má normálu v bodě A:

$$\vec{n}_1 = \left(\frac{\partial f(A)}{\partial x}, \frac{\partial f(A)}{\partial y}, -1\right)$$

$$\vec{n}_2 = \left(-\frac{\partial f(A)}{\partial x}, -\frac{\partial f(A)}{\partial y}, 1\right)$$

orientace plochy: výběr normály

Plošný integrál 1. druhu: Je-li S plocha zadaná jako $z=f(x,y)$ na uzavřené množině D, definujeme integrál funkce $F=F(x,y,z)$ na ploše S jako

$$\iint_S F(x,y,z)\,\underline{dS} = \iint_D F(x,y,f(x,y))\,\underline{\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dx\,dy}$$

Využití plošného integrálu: plocha $S: z=f(x,y)$ s plošnou hustotou $h=h(x,y,z)$

Povrch (obsah) plochy S: $P(S)=\iint_S 1\,dS$

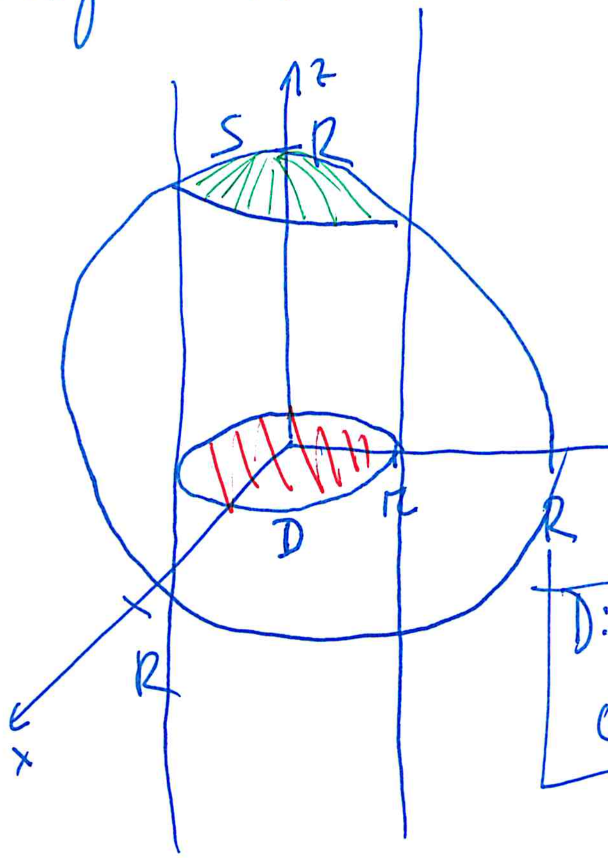
hmotnost: $m=\iint_S h(x,y,z)\,dS$

těžiště $x_T=\frac{1}{m}\iint_S x\cdot h(x,y,z)\,dS$ a analogicky y_T a z_T

moment setrvačnosti plochy S vzhledem k ose z

$$I_z=\iint_S \underbrace{(x^2+y^2)}_{r^2}\cdot\underbrace{h(x,y,z)\,dS}_{m}$$

Příklad: Spočítejte obsah kulové plochy (sféry) $x^2+y^2+z^2=R^2$ uvnitř válcové plochy $x^2+y^2=r^2$ pro $z\geq 0$, $R \geq r$



plocha: $z=\sqrt{R^2-x^2-y^2}$

$$\frac{\partial z}{\partial x}=\frac{-x}{\sqrt{R^2-x^2-y^2}}$$

$$\frac{\partial z}{\partial y}=\frac{-y}{\sqrt{R^2-x^2-y^2}}$$

$$\boxed{\begin{aligned} D:\ & 0\leq \rho \leq r \\ & 0 \leq \varphi \leq 2\pi \end{aligned}}$$

$$dS=\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy=\sqrt{1+\frac{x^2+y^2}{R^2-x^2-y^2}}\,dx\,dy=\frac{R}{\sqrt{R^2-x^2-y^2}}\,dx\,dy$$

$$P=\iint_S 1\,dS=\iint_D \frac{R}{\sqrt{R^2-x^2-y^2}}\,dx\,dy=\iint_D \frac{R\rho}{\sqrt{R^2-\rho^2}}\,d\rho\,d\varphi=$$

$$=2\pi R\int_0^r \frac{\rho}{\sqrt{R^2-\rho^2}}\,d\rho=-\pi R\int_{R^2}^{R^2-r^2}\frac{dt}{\sqrt{t}}=-\pi R\,2\left[\sqrt{t}\right]_{R^2}^{R^2-r^2}=$$

$t=R^2-\rho^2$

$dt=-2\rho\,d\rho$

$-\frac{1}{2}dt=\rho\,d\rho$

$$=\boxed{2\pi R\left(R-\sqrt{R^2-r^2}\right)}$$

Plošný integrál 2. druhu: Buď S omezená dvoustranná plocha orientovaná jednotkovým normálovým vektorem $\vec{n}$ a $\vec{a} = u(x,y,z)\vec{i} + v(x,y,z)\vec{j} + w(x,y,z)\vec{k}$ vektorové pole. Plošným integrálem 2. druhu pole $\vec{a}$ přes plochu S (tzv. tok pole $\vec{a}$ plochou S) definujeme jako integrál:

$$\iint_S u(x,y,z)\,dy\,dz + v(x,y,z)\,dx\,dz + w(x,y,z)\,dx\,dy \overset{\text{označení}}{=} \boxed{\iint_S \vec{a}\cdot\vec{n}\,dS}$$

Výpočet pl. int. 2. druhu: máme plochu $z = f(x,y)$ nad množinou D a pole

$$\vec{a} = u\cdot\vec{i} + v\cdot\vec{j} + w\cdot\vec{k}$$

1) potřebujeme jednotkovou normálu:

$$\vec{n}_{1p} = \frac{1}{\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}}\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\right)$$

nebo

$$\vec{n}_{2p} = \frac{1}{\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}}\left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right)$$

orientace nahoru

$$\iint_S \vec{a}\cdot\vec{n}\,dS = \iint_S (u,v,w)\,\frac{1}{\sqrt{}}\cdot\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\ \right)dS =$$

$$= \iint_D (u,v,w)\cdot\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},-1\right)\frac{1}{\cancel{\sqrt{}}}\,\cancel{\sqrt{}}\,dx\,dy$$

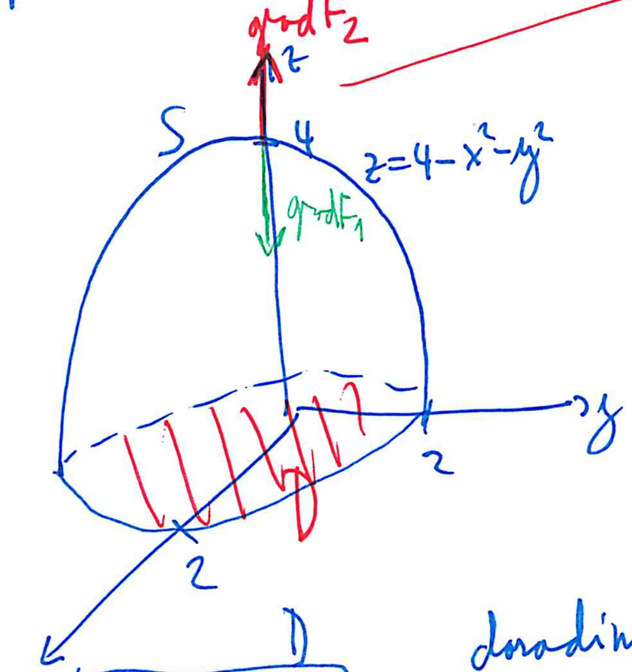
$$\downarrow$$

$$\boxed{\iint_S \vec{a}\cdot\vec{n}\,dS = \iint_D \vec{a}\cdot\operatorname{grad} F\,dx\,dy}$$, kde

S je plocha zadaná: $F:\ z - f(x,y) = 0$ nebo

$f(x,y) - z = 0$ (dle orientace)

Příklad: Vypočtěte tok pole $\vec{a} = x\vec{i} + y\vec{j} + z\vec{k}$ (6) plochou S, kterou tvoří část paraboloidu $z = 4 - x^2 - y^2$ pro $z \geq 0$ orientovaná normálou "mířící ven"



$F_1: \; 4 - x^2 - y^2 - z = 0$

$F_2: \; x^2 + y^2 + z - 4 = 0$

$\text{grad}\, F_1 = (-2x, -2y, -1)$

$\text{grad}\, F_2 = (2x, 2y, 1)$

dosadíme $x = 0, y = 0$ $\quad$ $\text{grad}\, F_1 = (0, 0, -1)$

$\text{grad}\, F_2 = (0, 0, 1)$

D

$$\boxed{\begin{array}{l} 0 \leq \rho \leq 2 \\ 0 \leq \varphi \leq 2\pi \end{array}}$$

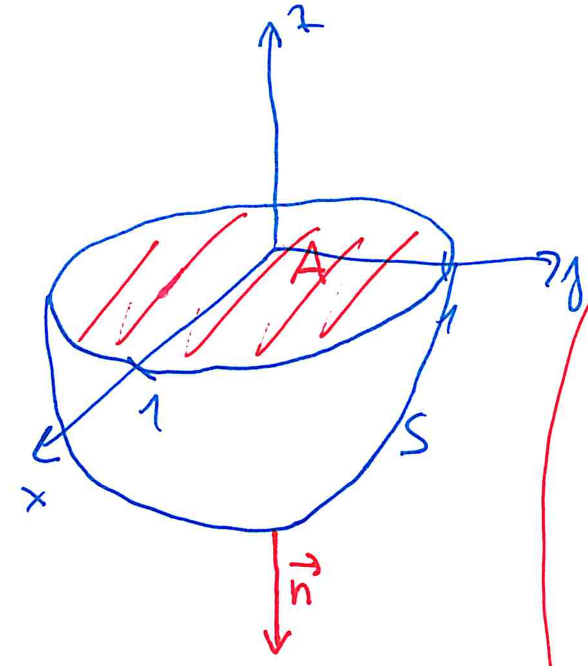
$$\iint_S \vec{a} \cdot \vec{n} \, dS = \iint_D \overset{\vec{a}}{(x, y, \overset{!}{z})} \cdot \overset{\text{grad} F_2}{(2x, 2y, 1)} \, dx\,dy =$$

$$= \iint_D (x, y, 4 - x^2 - y^2) \cdot (2x, 2y, 1) \, dx\,dy = \iint_D (2x^2 + 2y^2 + 4 - x^2 - y^2) \, dx\,dy =$$

$$= \iint_D (4 + x^2 + y^2) \, dx\,dy = \iint_D (4 + \rho^2) \cdot \rho \, d\rho\, d\varphi = \int_0^{2\pi} \left(\int_0^2 (4\rho + \rho^3) \, d\rho \right) d\varphi =$$

$$= 2\pi \left[2\rho^2 + \frac{\rho^4}{4} \right]_0^2 = 2\pi (8 + 4) = \boxed{24\pi}$$

Vypočtěte tok pole $\vec{a} = (-y, x, x^2y^2z)$ spodní polosférou $x^2+y^2+z^2=1$ orientovanou normálou $\vec{n}=(0,0,-1)$ pod bodem $[0,0]$



$$z = -\sqrt{1-x^2-y^2}$$

$$F: z + \sqrt{1-x^2-y^2} = 0$$

$$\operatorname{grad} F = \left(\frac{-x}{\sqrt{1-x^2-y^2}}, \frac{-y}{\sqrt{1-x^2-y^2}}, 1\right)$$

$\downarrow$

otočíme znaménko

$\downarrow$

$$\operatorname{grad} F = \left(\frac{x}{\sqrt{}}, \frac{y}{\sqrt{}}, -1\right)$$

$$\iint_S \vec{a}\cdot\vec{n}\, dS = \iint_A \left(-y, x, -x^2y^2\sqrt{1-x^2-y^2}\right)\cdot\left(\frac{x}{\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{1-x^2-y^2}}, -1\right) dx\,dy$$

$$= \iint_A x^2y^2\sqrt{1-x^2-y^2}\, dx\,dy$$

$$\iint_A x^2 y^2 \sqrt{1-x^2-y^2}\, dx\, dy = \iint_A \rho^2 \cos^2\varphi\, \rho^2 \sin^2\varphi \sqrt{1-\rho^2}\, \rho\, d\rho\, d\varphi =$$

$0 \le \varphi \le 2\pi \qquad x = \rho\cos\varphi$

$0 \le \rho \le 1 \qquad y = \rho\sin\varphi$

$$= \iint_A \underline{\rho^5 \sqrt{1-\rho^2}} \cos^2\varphi \sin^2\varphi\, \underline{d\rho}\, d\varphi = \frac{8}{105} \int_0^{2\pi} \cos^2\varphi \sin^2\varphi\, d\varphi = \frac{2\pi}{105}$$

$$\int_0^1 \rho^5 \sqrt{1-\rho^2}\, d\rho = -\frac{1}{2}\int_1^0 (1-t)^2 \sqrt{t}\, dt = \frac{1}{2}\int_0^1 \left(\sqrt{t} - 2t^{\frac{3}{2}} + t^{\frac{5}{2}}\right) dt =$$

$\rho^5 = \rho^4 \cdot \rho$

$t = 1-\rho^2$

$\rho^2 = 1-t$

$\rho^4 = (1-t)^2$

$dt = -2\rho\, d\rho$

$-\frac{1}{2} dt = \rho\, d\rho$

$$= \frac{1}{2}\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}} - 2\frac{t^{\frac{5}{2}}}{\frac{5}{2}} + \frac{t^{\frac{7}{2}}}{\frac{7}{2}}\right]_0^1 = \frac{1}{2}\left(\frac{2}{3} - \frac{4}{5} + \frac{2}{7}\right) =$$

$$= \frac{1}{2}\,\frac{70-84+30}{105} = \frac{8}{105}$$

$$\int_0^{2\pi} \cos^2\varphi \sin^2\varphi\, d\varphi = \frac{1}{4}\int_0^{2\pi} \sin^2 2\varphi\, d\varphi = \frac{1}{4}\int_0^{2\pi} \frac{1-\cos 4\varphi}{2}\, d\varphi = \frac{1}{8}\left[\varphi - \frac{1}{4}\sin 4\varphi\right]_0^{2\pi} =$$

$\cos^2\varphi - \sin^2\varphi = \cos 2\varphi$

$1 - 2\sin^2\varphi = \cos 2\varphi \rightarrow \sin^2\varphi = \frac{-\cos 2\varphi + 1}{2}$

$$= \frac{1}{8}\cancel{\pi}\, 2\pi$$