OPRAVA: Je-li pravá strana ve tvaru

$$\vec{f}(x) = e^{\alpha x} \cdot \vec{Q}_r(x), \text{ kde } \vec{Q}_r \text{ je}$$

vektor, jehož složky jsou polynomy st. nejvýše r,
hledáme

$$\vec{y}_p = e^{\alpha x} \vec{R}_{r+k}(x), \text{ kde } \vec{R}_{r+k} \text{ je}$$

polynom, jehož složky jsou polynomy st. $r+k$, kde
k je násobnost α jako kořene char. rovnice
(místo $x^k \to$ celý polynom st. k)

$$\vec{y}\,' = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \vec{y} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2x}$$

1) $\vec{y}\,' = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \vec{y}$

$$\det \begin{bmatrix} 2-\lambda & 0 \\ 1 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 = 0$$

$$\lambda_{1,2} = 2$$

vl. vektor pro $\lambda = 2$

$$A - \lambda E = A - 2E = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad x = 0 \quad \vec{u}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{y}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2x}$$

zob.. vl. vektor.

řešení soustavy s maticí $(A-\lambda E)^2 = (A-2E)^2$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

zobecněný vektor může být cokoliv,

ale nesmí to být vl. vektor. např. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

pro $\vec{u}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\vec{y}_2 = e^{\lambda x}\left(E + (A-\lambda E)\cdot x\right)\cdot \vec{u}_2$$

$$\vec{y}_2 = e^{2x}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\cdot x\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$

$$= e^{2x}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = e^{2x}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) =$$

$$= e^{2x}\begin{bmatrix} 1 \\ x \end{bmatrix} \qquad \vec{y}_H = C_1 e^{2x}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + C_2 e^{2x}\begin{bmatrix} 1 \\ x \end{bmatrix}$$

2) hledáme ~~vše~~ partikulární řešení nehomogenní s.

$\alpha = 2 \quad \beta = 0 \qquad \alpha + i\beta = 2 + 0i = 2 \rightarrow k = 2$

TO JE DVOJNÁSOBNÉ VL. číslo

$$\vec{y}_p = \begin{bmatrix} Ax^2 + Bx + C \\ Dx^2 + Ex + F \end{bmatrix} e^{2x}$$

$$\vec{y}_p{}' = \begin{bmatrix} 2Ax + B \\ 2Dx + E \end{bmatrix} e^{2x} + 2\begin{bmatrix} Ax^2 + Bx + C \\ Dx^2 + Ex + F \end{bmatrix} e^{2x}$$

4

a dosadíme do soustavy

$$\vec{y}' = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \vec{y} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2x}$$

$$\cancel{e^{2x}} \begin{bmatrix} 2Ax^2 + 2Bx + 2C + 2Ax + B \\ 2Dx^2 + 2Ex + 2F + 2Dx + E \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} Ax^2 + Bx + C \\ Dx^2 + Ex + F \end{bmatrix} \cancel{e^{2x}} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cancel{e^{2x}}$$

1. složka

$$2Ax^2 + 2Bx + 2C + 2Ax + B = 2(Ax^2 + Bx + C) + 1$$

2. složka

$$2Dx^2 + 2Ex + 2F + 2Dx + E = Ax^2 + Bx + C + 2(Dx^2 + Ex + F) + 1$$

1. složka u x^2: $2A = 2A$

u x: $\cancel{2B} + 2A = \cancel{2B} \rightarrow A = 0$

u x^0: $B + \cancel{2C} = \cancel{2C} + 1 \rightarrow B = 1$

2. složka u x^2: $\cancel{2D} = A + \cancel{2D}$

u x: $2D + \cancel{2E} = B + \cancel{2E} \rightarrow D = \frac{1}{2}$

u x^0: $E + \cancel{2F} = C + \cancel{2F} + 1$

$C = s \ (s \in \mathbb{R})$

$E = s + 1$

$F = r \ (r \in \mathbb{R})$

$$\vec{y}_p = \begin{bmatrix} x + A_2 \\ \frac{1}{2}x^2 + (A_2+1)x + r \end{bmatrix} e^{2x} \qquad A_2, r \in \mathbb{R}$$

volíme $A_2 = r = 0$

(A_2 je vlastne c_2
r je vlastne c_1)

$$\vec{y}_p = \begin{bmatrix} x \\ \frac{1}{2}x^2 + x \end{bmatrix} e^{2x}$$

$$\vec{y} = y_H + y_p = c_1 e^{2x} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 e^{2x} \begin{bmatrix} 1 \\ x \end{bmatrix} + \begin{bmatrix} x \\ \frac{1}{2}x^2 + x \end{bmatrix} e^{2x}$$

$$\vec{y}' = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 0 & -2 \\ 3 & 4 & 5 \end{bmatrix} \vec{y} + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} e^{x}$$

1) $$\vec{y}' = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 0 & -2 \\ 3 & 4 & 5 \end{bmatrix} \vec{y}$$

$$\det \begin{bmatrix} 1-\lambda & -2 & -1 \\ -2 & -\lambda & -2 \\ 3 & 4 & 5-\lambda \end{bmatrix} = (1-\lambda)(-\lambda)(5-\lambda) + 12 + 8 - \big(3\lambda - 8(1-\lambda) + 4(5-\lambda)\big) =$$

$$= -\lambda^3 + 6\lambda^2 - 5\lambda + 20 - (7\lambda + 12) = -\lambda^3 + 6\lambda^2 - 12\lambda + 8 =$$

$$= -(\lambda - 2)^3 = 0 \qquad \lambda_{1,2,3} = 2$$

hledáme vl. vektor pro $\lambda = 2$:

$$A - 2E = \begin{bmatrix} -1 & -2 & -1 \\ -2 & -2 & -2 \\ 3 & 4 & 3 \end{bmatrix} \begin{matrix} -2 \;\; 3 \\ \leftarrow \\ \leftarrow \end{matrix} \sim \begin{bmatrix} -1 & -2 & -1 \\ 0 & 2 & 0 \\ \cancel{0} & \cancel{2} & \cancel{0} \end{bmatrix} \begin{matrix} (-1) \\ \\ \\ \end{matrix}$$

$x + 2y + z = 0$ $\qquad z = t \; (t \in \mathbb{R})$ $\qquad \vec{u}_1 = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $\quad t = 1 \quad \vec{u}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$2y = 0$ $\qquad x = -t$

$$\vec{y}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{2x}$$

zobecněný vl. vektor:

řešíme soustavu s maticí $(A-2E)^2$

$$\underbrace{\begin{bmatrix} -1 & -2 & -1 \\ -2 & -2 & -2 \\ 3 & 4 & 3 \end{bmatrix}}_{A-2E} \underbrace{\begin{bmatrix} -1 & -2 & -1 \\ -2 & -2 & -2 \\ 3 & 4 & 3 \end{bmatrix}}_{A-2E} = \underbrace{\begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \\ -2 & -2 & -2 \end{bmatrix}}_{(A-2E)^2} \left(\tfrac{1}{2}\right)$$

$$\downarrow$$

$$x + y + z = 0$$

$$\downarrow$$

$z = s \ (s \in \mathbb{R})$

$y = r \ (r \in \mathbb{R})$

$x = -r - s$

$$\vec{u}_2 = \begin{bmatrix} -r-s \\ r \\ s \end{bmatrix} = r \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

TOTO JE ALE VL. VEKTOR

$$\vec{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{y}_2 = e^{2x} \left(E + (A-2E) \cdot x \right) \cdot \vec{u}_2$$

$$\vec{y}_2 = e^{2x} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + x \begin{bmatrix} -1 & -2 & -1 \\ -2 & -2 & -2 \\ 3 & 4 & 3 \end{bmatrix} \right) \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{y}_2 = e^{2x}\left(\begin{bmatrix}-1\\1\\0\end{bmatrix} + x\begin{bmatrix}-1\\0\\1\end{bmatrix}\right) = e^{2x}\begin{bmatrix}-1-x\\1\\x\end{bmatrix}$$

Potřebujeme ještě jedno řešení

řešíme soustavu s maticí ~~A_2~~ ~~A_2~~ $(A-2E)^3$

$$(A-2E)^3 = \underbrace{\begin{bmatrix}2&2&2\\0&0&0\\-2&-2&-2\end{bmatrix}}_{(A-2E)^2} \cdot \underbrace{\begin{bmatrix}-1&-2&-1\\-2&-2&-2\\3&4&3\end{bmatrix}}_{A-2E} = \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

$\vec{u}_3$ může být cokoliv, ale $(A-2E)^2 \cdot \vec{u}_3 \neq \vec{o}$

$\vec{u}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$ (náhodně zvoleno)

$$\vec{y}_3 = e^{2x}\left(E + (A-2E)x + \frac{1}{2}(A-2E)^2x^2\right)\cdot\vec{u}_3$$

$$\vec{y}_3 = e^{2x}\left(\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} + x\begin{bmatrix}-1&-2&-1\\-2&-2&-2\\3&4&3\end{bmatrix} + \frac{x^2}{2}\begin{bmatrix}2&2&2\\0&0&0\\-2&-2&-2\end{bmatrix}\right)\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\vec{y}_3 = e^{2x}\left(\begin{bmatrix}0\\0\\1\end{bmatrix} + x\begin{bmatrix}-1\\-2\\3\end{bmatrix} + \frac{x^2}{2}\begin{bmatrix}2\\0\\-2\end{bmatrix}\right) = e^{2x}\begin{bmatrix}-x+x^2\\-2x\\1+3x-x^2\end{bmatrix}$$

$$\vec{y}_H = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{2x} + c_2 \begin{bmatrix} -1-x \\ 1 \\ x \end{bmatrix} e^{2x} + c_3 \begin{bmatrix} x^2-x \\ -2x \\ 1+3x-x^2 \end{bmatrix} e^{2x}$$ 9

2) hledáme partikulární řešení

$$\vec{y}\,' = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 0 & -2 \\ 3 & 4 & 5 \end{bmatrix} \vec{y} + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} e^{x}$$

$\alpha = 1,\ \beta = 0 \quad \alpha + i\beta = 1$ není kořenem ch.r. $\to k = 0$

$$\vec{y}_p = \begin{bmatrix} A \\ B \\ C \end{bmatrix} e^{x} \qquad \vec{y}_p\,' = \begin{bmatrix} A \\ B \\ C \end{bmatrix} e^{x}$$

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} \cancel{e^{x}} = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 0 & -2 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} \cancel{e^{x}} + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \cancel{e^{x}}$$

1.ř. $A = A - 2B - C + 1$

2.ř. $B = -2A - 2C + 1$

3.ř. $C = 3A + 4B + 5C + 2$

$+2B+C=1$

$2A+B+2C=1$

$-3A-4B-4C=2$

$$\left[\begin{array}{ccc|c}2&1&2&1\\0&2&1&1\\-3&-4&-4&2\end{array}\right]\begin{array}{c}3\\ \\2\end{array}\sim\left[\begin{array}{ccc|c}2&1&2&1\\0&2&1&1\\0&-5&-2&7\end{array}\right]\begin{array}{c}\\5\\2\end{array}\sim\left[\begin{array}{ccc|c}2&1&2&1\\0&2&1&1\\0&0&1&19\end{array}\right]$$

$2A+B+2C=1 \qquad 2A=1-B-2C=1+9-38=-28\to A=-14$

$2B+C=1\to\quad 2B=4-C=-18\to B=-9$

$C=19$

$$\vec{y}_p=\begin{bmatrix}-14\\-9\\19\end{bmatrix}e^x$$

$$\vec{y}=\vec{y}_H+\vec{y}_p=C_1\begin{bmatrix}-1\\0\\1\end{bmatrix}e^{2x}+C_2\begin{bmatrix}-1-x\\1\\x\end{bmatrix}e^{2x}+C_3\begin{bmatrix}x^2-x\\-2x\\1+3x-x^2\end{bmatrix}e^{2x}+\begin{bmatrix}-14\\-9\\19\end{bmatrix}e^x$$

$$\vec{y}\,' = \begin{bmatrix} 5 & 0 & 4 \\ -3 & 3 & -7 \\ -2 & 0 & -1 \end{bmatrix} \vec{y} + \begin{bmatrix} x \\ 1 \\ 0 \end{bmatrix}$$

matice má dvojnásobné vl. číslo $\lambda_{1,2} = 3$ a vl. vektor $\vec{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

a vl. číslo $\lambda_3 = 1$ a vl. vektor $\vec{u}_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

$\vec{y}_1 = e^{3x} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{y}_2 = e^{x} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ zbývá najít zob. vl. vektor pro $\lambda = 3$

$\alpha = 0$, $\beta = 0$, $\alpha + i\beta = 0$ (není kořen ch. r.) $\to k = 0$

$\vec{y}_p = \begin{bmatrix} Ax + B \\ Cx + D \\ Ex + F \end{bmatrix}$ $\vec{y}_p\,' = \begin{bmatrix} A \\ C \\ E \end{bmatrix}$

a dosadíme do soustavy

$$\begin{bmatrix} A \\ C \\ E \end{bmatrix} = \begin{bmatrix} 5 & 0 & 4 \\ -3 & 3 & -7 \\ -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} Ax+B \\ Cx+D \\ Ex+F \end{bmatrix} + \begin{bmatrix} x \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{y}_p = \begin{bmatrix} \frac{1}{3}x + \frac{7}{9} \\ -\frac{11}{9}x - \frac{55}{27} \\ -\frac{2}{3}x - \frac{8}{9} \end{bmatrix}$$

$$A = 5(Ax+B) + 4(Ex+F) + x$$
$$C = -3(Ax+B) + 3(Cx+D) - 7(Ex+F) + 1$$
$$E = -2(Ax+B) - (Ex+F)$$

1.r. $u\,x: \; 0 = 5A + 4E + 1$

$u\,x^0: \; A = 5B + 4F$

2.r. $u\,x: \; 0 = -3A + 3C - 7E$

$u\,x^0: \; C = -3B + 3D - 7F + 1$

3.r. $u\,x: \; 0 = -2A - E$

$u\,x^0: \; E = -2B - F$

$\times 4$

$$-3A + 1 = 0 \rightarrow \boxed{A = \frac{1}{3} \quad E = -\frac{2}{3}}$$

$$1 = 3C + \frac{14}{3} \rightarrow 3C = \frac{-11}{3} \qquad \boxed{C = -\frac{11}{9}}$$

$$-\frac{11}{9} = \frac{-21}{9} + 3D + \frac{56}{9} + \frac{9}{9}$$

$$3D = \frac{-11+21-56-9}{9} = \boxed{\frac{-55}{9}}$$

$$\frac{1}{3} = 5B + 4F$$

$$-\frac{2}{3} = -2B - F \quad / \cdot 4$$

$$-\frac{7}{3} = -3B \longrightarrow \boxed{B = \frac{7}{9} \qquad F = \frac{2}{3} - 2B = \frac{2}{3} - \frac{14}{9} = -\frac{8}{9}}$$