SOUSTAVY OBYČEJNÝCH LINEÁRNÍCH DIFERENCIÁLNÍCH ROVNIC

Příklad 1

a) $$\vec{y}\,' = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix} \vec{y}$$

$$\begin{vmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)(-1-\lambda) + 3\cdot 4 + 2\cdot 1 - 4\cdot 2\cdot(2-\lambda) + (1-\lambda) + 3(-1-\lambda) =$$

$$= -(1-\lambda)(2-\lambda)(1+\lambda) + 12 + 2 - 8(2-\lambda) + 1 - \lambda - 3 - 3\lambda =$$

$$= -(1-\lambda)(2-\lambda)(1+\lambda) - 4 + 4\lambda = -(1-\lambda)(2-\lambda)(1+\lambda) - 4(1-\lambda) =$$

$$= (1-\lambda)\left[-(2-\lambda)(1+\lambda) - 4\right] = (1-\lambda)\cdot(-2-\lambda+\lambda^2-4) =$$

$$= (1-\lambda)(\lambda^2-\lambda-6) = (1-\lambda)(\lambda-3)(\lambda+2)$$

$\lambda_1 = 1,\ \lambda_2 = 3,\ \lambda_3 = -2$

$\lambda_1 = 1:$ $$\begin{pmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{pmatrix} \vec{v}_1 = \vec{o}$$

$$\begin{pmatrix} 3 & 1 & -1 \\ 0 & -1 & 4 \\ 2 & 1 & -2 \end{pmatrix} \underset{3(3)-2(1)}{\sim} \begin{pmatrix} 3 & 1 & -1 \\ 0 & -1 & 4 \\ \cancel{0} & \cancel{1} & \cancel{-4} \end{pmatrix} \sim \begin{pmatrix} 3 & 1 & -1 \\ 0 & -1 & 4 \end{pmatrix}$$

$$\left.\begin{aligned} 3x + y - z &= 0 \\ -y + 4z &= 0 \end{aligned} \Rightarrow \begin{aligned} &z = t,\ t \in \mathbb{R} - \{0\} \\ &y = 4z = 4t \\ &x = \frac{z-y}{3} = -t \end{aligned}\right\} \Rightarrow \vec{v}_1 = t\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix},\ t=1:$$

$$\vec{v}_1 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$$

$\lambda_2 = 3:$ $$\begin{pmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{pmatrix} \vec{v}_2 = \vec{o}$$

$$\begin{pmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ \cancel{2} & \cancel{1} & \cancel{-4} \end{pmatrix} \underset{2(2)+3(1)}{\sim} \begin{pmatrix} -2 & -1 & 4 \\ 0 & -5 & 10 \end{pmatrix} \sim \begin{pmatrix} -2 & -1 & 4 \\ 0 & 1 & -2 \end{pmatrix}$$

$$-2x - y + 4z = 0$$
$$y - 2z = 0 \Rightarrow \left.\begin{array}{l} z = t,\ t \in \mathbb{R} - \{0\} \\ y = 2z = 2t \\ x = \frac{4z - y}{2} = t \end{array}\right\} \Rightarrow \vec{v}_2 = t\begin{pmatrix}1\\2\\1\end{pmatrix},\ t = 1:$$
$$\vec{v}_2 = \begin{pmatrix}1\\2\\1\end{pmatrix}$$

$\lambda_3 = -2$:

$$\begin{pmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \\ 2 & 1 & 1 \end{pmatrix} \cdot \vec{v}_3 = \vec{o}$$

$$\begin{pmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \\ 2 & 1 & 1 \end{pmatrix} \begin{array}{c} \\ (2)-(1) \\ \sim \\ 3(3)-2(1) \end{array} \begin{pmatrix} 3 & -1 & 4 \\ 0 & 5 & -5 \\ \cancel{0} & \cancel{5} & \cancel{-5} \end{pmatrix} \sim \begin{pmatrix} 3 & -1 & 4 \\ 0 & 1 & -1 \end{pmatrix}$$

$$3x - y + 4z = 0$$
$$y - z = 0 \Rightarrow \left.\begin{array}{l} z = t,\ t \in \mathbb{R} - \{0\} \\ y = z = t \\ x = \frac{y - 4z}{3} = -t \end{array}\right\} \Rightarrow \vec{v}_3 = t\begin{pmatrix}-1\\1\\1\end{pmatrix},\ t = 1:$$
$$\vec{v}_3 = \begin{pmatrix}-1\\1\\1\end{pmatrix}$$

$$\vec{y}_1 = e^{x}\begin{pmatrix}-1\\4\\1\end{pmatrix},\ \vec{y}_2 = e^{3x}\begin{pmatrix}1\\2\\1\end{pmatrix},\ \vec{y}_3 = e^{-2x}\begin{pmatrix}-1\\1\\1\end{pmatrix}$$

$$\underline{\vec{y} = c_1 e^{x}\begin{pmatrix}-1\\4\\1\end{pmatrix} + c_2 e^{3x}\begin{pmatrix}1\\2\\1\end{pmatrix} + c_3 e^{-2x}\begin{pmatrix}-1\\1\\1\end{pmatrix}}\ ;\ c_1, c_2, c_3 \in \mathbb{R}$$

b) $\vec{y}\,' = \begin{pmatrix} -1 & 2 & -3 \\ 5 & -3 & 2 \\ 1 & 0 & -3 \end{pmatrix} \vec{y}\ ,\ \vec{y}(0) = \begin{pmatrix}4\\14\\3\end{pmatrix}$

$$\begin{vmatrix} -1-\lambda & 2 & -3 \\ 5 & -3-\lambda & 2 \\ 1 & 0 & -3-\lambda \end{vmatrix} = (-1-\lambda)(-3-\lambda)^2 + 4 + 3(-3-\lambda) - 2\cdot 5\cdot(-3-\lambda) =$$
$$= -(1+\lambda)(3+\lambda)^2 + 4 - 9 - 3\lambda + 30 + 10\lambda =$$
$$= -(1+\lambda)(9 + 6\lambda + \lambda^2) + 25 + 7\lambda = -(9 + 6\lambda + \lambda^2 + 9\lambda + 6\lambda^2 + \lambda^3) +$$
$$+ 25 + 7\lambda = -\lambda^3 - 7\lambda^2 - 8\lambda + 16$$

$\lambda = 1 : \ -1 - 7 - 8 + 16 = 0 \Rightarrow 1$ je koren

$(-\lambda^3 - 7\lambda^2 - 8\lambda + 16) : (\lambda - 1) = -\lambda^2 - 8\lambda - 16 = -(\lambda^2 + 8\lambda + 16) = -(\lambda + 4$

$$
\begin{array}{l}
+\lambda^3 - \lambda^2 \\
\hline
-8\lambda^2 - 8\lambda + 16 \\
+8\lambda^2 - 8\lambda \\
\hline
-16\lambda + 16 \\
+16\lambda - 16 \\
\hline
0
\end{array}
$$

$\lambda_1 = 1, \ \lambda_{2,3} = -4$

$$\lambda_1 = 1 : \begin{pmatrix} -2 & 2 & -3 \\ 5 & -4 & 2 \\ 1 & 0 & -4 \end{pmatrix} \vec{v}_1 = \vec{0}$$

$$\begin{pmatrix} 1 & 0 & -4 \\ 5 & -4 & 2 \\ -2 & 2 & -3 \end{pmatrix} \underset{(3)+2(1)}{\overset{(2)-5(1)}{\sim}} \begin{pmatrix} 1 & 0 & -4 \\ 0 & -4 & 22 \\ 0 & 2 & -11 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -4 \\ 0 & -2 & 11 \\ \cancel{0} & \cancel{2} & \cancel{-11} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -4 \\ 0 & -2 & 11 \end{pmatrix}$$

$$\left.\begin{array}{l} x - 4z = 0 \\ -2y + 11z = 0 \end{array} \Rightarrow \begin{array}{l} z = t, \ t \in \mathbb{R} - \{0\} \\ y = \frac{11z}{2} = \frac{11}{2}t \\ x = 4z = 4t \end{array}\right\} \Rightarrow \vec{v}_1 = t\begin{pmatrix} 4 \\ \frac{11}{2} \\ 1 \end{pmatrix}, \ t = 2 :$$

$$\vec{v}_1 = \begin{pmatrix} 8 \\ 11 \\ 2 \end{pmatrix}$$

$$\lambda_2 = -4 : \begin{pmatrix} 3 & 2 & -3 \\ 5 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \vec{v}_2 = \vec{0}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 5 & 1 & 2 \\ 3 & 2 & -3 \end{pmatrix} \underset{(3)-3(1)}{\overset{(2)-5(1)}{\sim}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 2 & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ \cancel{0} & \cancel{1} & \cancel{-3} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \end{pmatrix}$$

$$\left.\begin{array}{l} x + z = 0 \\ y - 3z = 0 \end{array} \Rightarrow \begin{array}{l} z = t, \ t \in \mathbb{R} - \{0\} \\ y = 3z = 3t \\ x = -z = -t \end{array}\right\} \Rightarrow \vec{v}_2 = t\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}, \ t = 1 :$$

$$\vec{v}_2 = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$$

$$\lambda_3 = -4: \quad (A-\lambda E)^2 = \begin{pmatrix} 3 & 2 & -3 \\ 5 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 2 & -3 \\ 5 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 16 & 8 & -8 \\ 22 & 11 & -11 \\ 4 & 2 & -2 \end{pmatrix}$$

hledáme zobecněný vlastní vektor:

$$\begin{pmatrix} 16 & 8 & -8 \\ 22 & 11 & -11 \\ 4 & 2 & -2 \end{pmatrix} \vec{v}_3 = \vec{o}$$

$$\begin{pmatrix} 2 & 1 & -1 \\ \cancel{2} & \cancel{1} & \cancel{-1} \\ \cancel{2} & \cancel{1} & \cancel{-1} \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & -1 \end{pmatrix} \Rightarrow 2x + y - z = 0$$

$$\Rightarrow \left.\begin{aligned} x &= t, \ t \in \mathbb{R} - \{0\} \\ y &= s, \ s \in \mathbb{R} - \{0\} \\ z &= 2x + y = 2t + s \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \vec{v}_3 = \begin{pmatrix} t \\ s \\ 2t+s \end{pmatrix}, \quad t = 0, s = 1 \Rightarrow$$

$$\Rightarrow \vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$(A-\lambda E)\vec{v}_3 \overset{?}{\neq} \vec{o}: \quad \begin{pmatrix} 3 & 2 & -3 \\ 5 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{y}_1 = e^{x} \begin{pmatrix} 8 \\ 11 \\ 2 \end{pmatrix}, \ \vec{y}_2 = e^{-4x} \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}, \ \vec{y}_3 = e^{-4x} \left[E + x(A-\lambda E) \right] \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} =$$

$$= e^{-4x} \begin{pmatrix} 1+3x & 2x & -3x \\ 5x & x+1 & 2x \\ 1x & 0 & x+1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = e^{-4x} \begin{pmatrix} -x \\ 3x+1 \\ x+1 \end{pmatrix}$$

$$\vec{y} = c_1 e^{x} \begin{pmatrix} 8 \\ 11 \\ 2 \end{pmatrix} + c_2 e^{-4x} \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} + c_3 e^{-4x} \begin{pmatrix} -x \\ 3x+1 \\ x+1 \end{pmatrix}$$

dopočítáme c_1, c_2, c_3 z počáteční podmínky:

$$\vec{y}(0) = \begin{pmatrix} 7 \\ 14 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} 8 \\ 11 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$7 = 8c_1 - c_2 \quad \Rightarrow c_2 = 8c_1 - 7$$
$$14 = 11c_1 + 3c_2 + c_3$$
$$3 = 2c_1 + c_2 + c_3$$

$$14 = 11c_1 + 24c_1 - 21 + c_3$$
$$3 = 2c_1 + 8c_1 - 7 + c_3$$

$$35 = 35c_1 + c_3$$
$$10 = 10c_1 + c_3 \quad \ominus$$

$$25 = 25c_1 \Rightarrow c_1 = 1$$
$$c_3 = 10 - 10c_1 = 0$$
$$c_2 = 8c_1 - 7 = 1$$

$$\vec{y} = e^{x}\begin{pmatrix} 8 \\ 11 \\ 2 \end{pmatrix} + e^{-4x}\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$$

c) $\vec{y}' = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}\vec{y}$

$$\begin{vmatrix} -\lambda & 0 & -2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix} = \lambda^2(2-\lambda) + 2\cdot 2\cdot(2-\lambda) = \lambda^2(2-\lambda) + 4(2-\lambda) =$$
$$= (\lambda^2+4)(2-\lambda)$$

$\lambda_1 = 2, \quad \lambda_{2,3} = \pm 2i$

$$\lambda_1 = 2: \quad \begin{pmatrix} -2 & 0 & -2 \\ \cancel{0} & \cancel{0} & \cancel{0} \\ 2 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\left.\begin{array}{l} x + z = 0 \\ -2z = 0 \Rightarrow z = 0 \\ \qquad\qquad x = -z = 0 \\ \qquad\qquad y = t,\ t \in \mathbb{R} - \{0\} \end{array}\right\} \Rightarrow \vec{v}_1 = t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ t = 1:$$
$$\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_2 = 2i: \quad \begin{pmatrix} -2i & 0 & -2 \\ 0 & 2-2i & 0 \\ 2 & 0 & -2i \end{pmatrix} \vec{v_2} = \vec{0}$$

$$\begin{pmatrix} i & 0 & 1 \\ 0 & 2-2i & 0 \\ 1 & 0 & -i \end{pmatrix} \underset{i(3)-(1)}{\sim} \begin{pmatrix} i & 0 & 1 \\ 0 & 2-2i & 0 \\ 0 & 0 & \underbrace{-i^2}_{-1}-1 \end{pmatrix} \sim \begin{pmatrix} i & 0 & 1 \\ 0 & 2-2i & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} \end{pmatrix} \sim \begin{pmatrix} i & 0 & 1 \\ 0 & 2-2i & 0 \end{pmatrix}$$

$$\left.\begin{aligned} & i\cdot x + z = 0 \\ & (2-2i)\cdot y = 0 \Rightarrow y = 0 \\ & \qquad x = t, \ t \in \mathbb{C}-\{0\} \\ & \qquad z = -ix = -it \end{aligned}\right\} \Rightarrow \vec{v_2} = t\begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix}, \ t = 1:$$

$$\vec{v_2} = \begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix}$$

$$\lambda_3 = -2i: \quad \vec{v_3} = \overline{\vec{v_2}} = \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$$

$$\vec{y_1} = e^{2x}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \vec{y_2} = e^{2ix}\begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix}, \ \vec{y_3} = e^{-2ix}\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}$$

Euler. vzlah: $e^{\lambda x} = e^{(\alpha+i\beta)x} = e^{\alpha x}\cdot(\cos\beta x + i\sin\beta x)$

$$\frac{\vec{y_2}+\vec{y_3}}{2} = \frac{1}{2}\left[e^{2ix}\begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix} + e^{-2ix}\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}\right] =$$

$$= \frac{1}{2}\left[(\cos 2x + i\sin 2x)\cdot\begin{pmatrix} 1 \\ 0 \\ -i \end{pmatrix} + (\cos(-2x) + i\sin(-2x)\cdot\begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}\right] =$$

$$= \frac{1}{2}\left\{(\cos 2x + i\sin 2x)\cdot\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}\right] + \right.$$

$$\left. + (\cos 2x - i\sin 2x)\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - i\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}\right]\right\} =$$

$$= \frac{1}{2}\left[\cos 2x\cdot\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + i\cdot\cos 2x\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + i\cdot\sin 2x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \sin 2x\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} + \right.$$

$$\left. + \cos 2x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - i\cos 2x\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} - i\sin 2x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \sin 2x\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}\right] =$$

$$= \cos 2x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \sin 2x\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \mathrm{Re}(y_2)$$

$$\frac{y_2 - y_3}{2i} = \frac{1}{2i}\left[2i\cos 2x\begin{pmatrix}0\\0\\-1\end{pmatrix} + 2i\sin 2x\begin{pmatrix}1\\0\\0\end{pmatrix}\right] =$$

$$= \cos 2x\begin{pmatrix}0\\0\\-1\end{pmatrix} + \sin 2x\begin{pmatrix}1\\0\\0\end{pmatrix} = \mathrm{Im}(y_2)$$

$$\underline{\vec{y} = c_1 e^{2x}\begin{pmatrix}0\\1\\0\end{pmatrix} + c_3\left[\cos 2x\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} - \sin 2x\cdot\begin{pmatrix}0\\0\\-1\end{pmatrix}\right]} +$$

$$\underline{+ c_3\left[\cos 2x\begin{pmatrix}0\\0\\-1\end{pmatrix} + \sin 2x\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}\right]}$$

<u>Příklad 2</u>

a) $\vec{y}\,' = \begin{pmatrix}2&0&0\\1&0&2\\1&0&2\end{pmatrix}\vec{y} + \begin{pmatrix}3\\2\\1\end{pmatrix}e^{-x}$, $\vec{y}(0) = \begin{pmatrix}2\\3\\1\end{pmatrix}$

$$\begin{vmatrix}2-\lambda&0&0\\1&-\lambda&2\\1&0&2-\lambda\end{vmatrix} = -\lambda(2-\lambda)^2$$

$\lambda_1 = 0$, $\lambda_{2,3} = 2$

$\lambda_1 = 0$: $\begin{pmatrix}2&0&0\\1&0&2\\1&0&2\end{pmatrix}\vec{v}_1 = \vec{o}$

$$\begin{pmatrix}2&0&0\\1&0&2\\\cancel{1}&\cancel{0}&\cancel{2}\end{pmatrix} \underset{2(2)-(1)}{\sim} \begin{pmatrix}2&0&0\\0&0&4\end{pmatrix} \sim \begin{pmatrix}1&0&0\\0&0&1\end{pmatrix}$$

$$\left.\begin{aligned}x &= 0\\z &= 0\\y &= t,\ t \in \mathbb{R}-\{0\}\end{aligned}\right\} \Rightarrow \vec{v}_1 = t\begin{pmatrix}0\\1\\0\end{pmatrix},\ t = 1:$$

$$\vec{v}_1 = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$\lambda_2 = 2$: $\begin{pmatrix}0&0&0\\1&-2&2\\1&0&0\end{pmatrix}\vec{v}_2 = \vec{o}$

$$\begin{pmatrix}1&0&0\\1&-2&2\\\cancel{0}&\cancel{0}&\cancel{0}\end{pmatrix} \sim \begin{pmatrix}1&0&0\\0&-2&2\end{pmatrix} \sim \begin{pmatrix}1&0&0\\0&1&-1\end{pmatrix}$$

$$\left.\begin{aligned} &x = 0 \\ &y - z = 0 \Rightarrow z = t,\ t \in \mathbb{R} - \{0\} \\ &\qquad y = z = t \end{aligned}\right\} \Rightarrow \vec{v}_2 = t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},\ t = 1:$$

$$\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$(A - \lambda E)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 2 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 2 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & -4 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & -4 \\ 0 & 0 & 0 \end{pmatrix}\vec{v}_3 = \vec{o} \Rightarrow \left.\begin{aligned} &4y - 4z = 0 \\ &y - z = 0 \Rightarrow z = t,\ t \in \mathbb{R} \\ &\qquad y = z = t \\ &\qquad x = s,\ s \in \mathbb{R} \end{aligned}\right\} \Rightarrow \vec{v}_3 = \begin{pmatrix} s \\ t \\ t \end{pmatrix}$$

$$s = 1,\ t = 0:\ \vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

ale $(A - \lambda E)\vec{v}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 2 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \neq \vec{o}$

$$\vec{y}_1 = e^{0x}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\quad \vec{y}_2 = e^{2x}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},\quad \vec{y}_3 = e^{2x}\left[E + x(A - \lambda E)\right] \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} =$$

$$= e^{2x}\begin{pmatrix} 1 & 0 & 0 \\ x & 1-2x & 2x \\ x & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = e^{2x}\begin{pmatrix} 1 \\ x \\ x \end{pmatrix}$$

$$\underline{\vec{y}_H = c_1\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{2x}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{2x}\begin{pmatrix} 1 \\ x \\ x \end{pmatrix}}$$

$\lambda = -1$ je kořenem char. polynomu matice? NE

a složky vektoru $\vec{b}$ jsou polynomy stupně nejvýše 0 $\Rightarrow$

$$\Rightarrow \vec{y}_P = e^{-x} \cdot \begin{pmatrix} A \\ B \\ C \end{pmatrix},\quad \vec{y}_P' = -e^{-x}\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} -A \\ -B \\ -C \end{pmatrix} e^{-x}$$

$$\begin{pmatrix} -A \\ -B \\ -C \end{pmatrix} e^{-x} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} A \\ B \\ C \end{pmatrix} \cdot e^{-x} + \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} e^{-x} \qquad / : e^{-x} \neq 0 \quad \forall x \in \mathbb{R}$$

$$\begin{pmatrix} -A \\ -B \\ -C \end{pmatrix} = \begin{pmatrix} 2A \\ A+2C \\ A+2C \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$-A = 2A + 3$$
$$-B = A + 2C + 2$$
$$-C = A + 2C + 1$$

$$-3A = 3 \Rightarrow \boxed{A = -1}$$
$$-A - B - 2C = 2 \Rightarrow 1 - B - 2C = 2 \Rightarrow B + 2C = -1 \Rightarrow \boxed{B =} -1 - 2C = \boxed{-1}$$
$$-A - 3C = 1 \Rightarrow \boxed{C =} \frac{A+1}{-3} = \boxed{0}$$

$$\vec{y}_P = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} e^{-x}$$

$$\vec{y} = c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{2x} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{2x} \begin{pmatrix} 1 \\ x \\ x \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} e^{-x}$$

$$\vec{y}(0) = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$$

$$2 = c_3 - 1 \Rightarrow \boxed{c_3 = 3}$$
$$3 = c_1 + c_2 - 1 = \boxed{c_1 =} 4 - c_2 = \boxed{3}$$
$$\boxed{1 = c_2}$$

$$\vec{y} = 3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + e^{2x} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 3e^{2x} \begin{pmatrix} 1 \\ x \\ x \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} e^{-x}$$

b) $\vec{y}\,' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \vec{y} + \begin{pmatrix} -x^{2} \\ 2x \end{pmatrix} \cdot e^{0x}$

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 1 = 3 - 4\lambda + \lambda^2 + 1 = \lambda^2 - 4\lambda + 4 = $$
$$= (\lambda - 2)^2$$

$\lambda_{1,2} = 2$

$\lambda_1 = 2: \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \vec{v}_1 = \vec{o}$

$$\begin{pmatrix} -1 & -1 \\ \cancel{1} & \cancel{1} \end{pmatrix} \sim (1\ 1) \Rightarrow x + y = 0$$
$$\left.\begin{aligned} y &= t,\ t \in \mathbb{R} - \{0\} \\ x &= -y = -t \end{aligned}\right\} \Rightarrow \vec{v}_1 = t\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
$$t = 1 \cdot \vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$\lambda_2 = 2: (A - \lambda E)^2 = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$0 \cdot x + 0 \cdot y = 0 \Rightarrow x, y$ cokoliv, aby nebyl násobkem $\vec{v}_1$: $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\vec{y}_1 = e^{2x}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\vec{y}_2 = e^{2x}\left[E + x(A - \lambda E)\right] \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = e^{2x}\begin{pmatrix} 1-x & -x \\ x & x+1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} =$

$$= e^{2x}\begin{pmatrix} 1-2x \\ 2x+1 \end{pmatrix}$$

$$\vec{y}_H = c_1 e^{2x}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{2x}\begin{pmatrix} -2x+1 \\ 2x+1 \end{pmatrix}$$

$\vec{b}(x) = \begin{pmatrix} -x^2 \\ 2x \end{pmatrix} = e^{0x} \cdot \begin{pmatrix} -x^2 \\ 2x \end{pmatrix}$ ... je 0 kořenem char. polynomu matice soustavy? NE a složky jsou polynomy stupně nejvýše 2 $\Rightarrow \vec{y}_p = \begin{pmatrix} Ax^2 + Bx + C \\ Dx^2 + Ex + F \end{pmatrix}$

$$\vec{y}_P' = \begin{pmatrix} 2Ax + B \\ 2Dx + E \end{pmatrix}$$

$$\begin{pmatrix} 2Ax + B \\ 2Dx + E \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} Ax^2 + Bx + C \\ Dx^2 + Ex + F \end{pmatrix} + \begin{pmatrix} -x^2 \\ 2x \end{pmatrix} =$$

$$= \begin{pmatrix} Ax^2 + Bx + C - Dx^2 - Ex - F \\ Ax^2 + Bx + C + 3Dx^2 + 3Ex + 3F \end{pmatrix} + \begin{pmatrix} -x^2 \\ 2x \end{pmatrix} =$$

$$= \begin{pmatrix} Ax^2 + Bx + C - Dx^2 - Ex - F - x^2 \\ Ax^2 + Bx + C + 3Dx^2 + 3Ex + 3F + 2x \end{pmatrix} \Longrightarrow$$

$$\Rightarrow \begin{array}{l} 1)\ 2Ax + B = Ax^2 + Bx + C - Dx^2 - Ex - F - x^2 \\ 2)\ 2Dx + E = Ax^2 + Bx + C + 3Dx^2 + 3Ex + 3F + 2x \end{array}$$

$$1) \begin{array}{l} x^2: \ 0 = A - D - 1 \\ x^1: \ 2A = B - E \\ x^0: \ B = C - F \end{array} \qquad 2) \begin{array}{l} x^2: \ 0 = A + 3D \\ x^1: \ 2D = B + 3E + 2 \\ x^0: \ E = C + 3F \end{array}$$

soustava 6 rovnic o 6 neznámých

její řešení je: $A = \frac{3}{4}$

$B = \frac{1}{2}$

$C = \frac{1}{8}$

$D = -\frac{1}{4}$

$E = -1$

$F = -\frac{3}{8}$

$$\underline{\vec{y}} = \vec{y}_H + \vec{y}_P = \underline{\underline{c_1 e^{2x} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{2x} \begin{pmatrix} -2x + 1 \\ 2x + 1 \end{pmatrix} + \begin{pmatrix} \frac{3}{4}x^2 + \frac{1}{2}x + \frac{1}{8} \\ -\frac{1}{4}x^2 - x - \frac{3}{8} \end{pmatrix}}},$$

$c_1, c_2 \in \mathbb{R}$

c) $\vec{y}\,' = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix} \vec{y} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2x}, \quad \vec{y}(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & 3-\lambda \end{vmatrix} = (2-\lambda)^2(3-\lambda) = 0 \iff \lambda = 2 \vee \lambda = 3$$

$$\lambda_{1,2} = 2, \ \lambda_3 = 3$$

$\underline{\lambda_1 = 2:}$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \cdot \vec{v}_1 = \vec{o}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{aligned} y + z &= 0 \Rightarrow y = 0 \\ z &= 0 \end{aligned}$$

x libovolné, tj: $x = t$, $t \in \mathbb{R} - \{0\}$

$$\vec{v}_1 = \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$t = 1: \ \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\underline{\lambda_2 = 2 \Rightarrow \text{hledáme zobecněný vl. vektor:}}$

$$(A - \lambda E)^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$(A - \lambda E)^2 \cdot \vec{v}_2 = \vec{o} \iff \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \cdot \vec{v}_2 = \vec{o}$$

$$\begin{pmatrix} 0 & 1 & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{1} & \cancel{1} \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \Rightarrow y + z = 0$$

$z = t$, $t \in \mathbb{R} - \{0\}$

$y = -z = -t$

x libovolné, $x = s$, $s \in \mathbb{R} - \{0\}$

$$\vec{v}_2 = \begin{pmatrix} s \\ -t \\ t \end{pmatrix}, \quad s = 0, t = 1: \ \vec{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

c) $\vec{y}\,' = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \vec{y} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-x}$

$$\begin{vmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda+1)(\lambda-3) = 0$$

$\Leftrightarrow \lambda_1 = -1 \,,\ \lambda_2 = 3$

$\underline{\lambda_1 = -1:}$

$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \vec{v}_1 = \vec{0}$

$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \overset{|:2}{\sim} \begin{pmatrix} 1 & 2 \\ \cancel{1} & \cancel{2} \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \end{pmatrix}$

$\Rightarrow x + 2y = 0$

$y = t$

$x = -2y = -2t \,,\ t \in \mathbb{R} - \{0\}$

$\vec{v}_1 = \begin{pmatrix} -2t \\ t \end{pmatrix} = t \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$t = 1: \ \vec{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\underline{\lambda_2 = 3:}$

$\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} \vec{v}_2 = \vec{0}$

$\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} \overset{|:(-2)}{\sim} \begin{pmatrix} 1 & -2 \\ \cancel{1} & \cancel{-2} \end{pmatrix} \sim \begin{pmatrix} 1 & -2 \end{pmatrix} \Rightarrow x - 2y = 0$

$y = t$

$x = 2y = 2t \,,\ t \in \mathbb{R} - \{0\}$

$\vec{v}_2 = \begin{pmatrix} 2t \\ t \end{pmatrix} = t \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$t = 1: \ \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\vec{y}_1 = e^{-x} \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} \,,\ \vec{y}_2 = e^{3x} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\underline{\vec{y}_H = c_1 e^{-x} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 e^{3x} \begin{pmatrix} 2 \\ 1 \end{pmatrix}} \ ; c_1, c_2 \in \mathbb{R}$

$$\vec{b}(x) = e^{-x}\begin{pmatrix}1\\1\end{pmatrix}$$

je -1 kořenem char. polynomu matice soustavy?

<u>ANO</u>, jednonásobný kořen

a složky vektoru $\begin{pmatrix}1\\1\end{pmatrix}$ jsou polynomy

stupně nejvýše 0 $\Rightarrow$ $\vec{y}_p = e^{-x}\cdot\begin{pmatrix}Ax+B\\Cx+D\end{pmatrix}$ polynomy stupně $(0+1)=1$

$$\vec{y}_p = \begin{pmatrix}e^{-x}(Ax+B)\\e^{-x}(Cx+D)\end{pmatrix}$$

$$\vec{y}_p\,' = \begin{pmatrix}-e^{-x}\cdot(Ax+B)+e^{-x}\cdot A\\-e^{-x}\cdot(Cx+D)+e^{-x}\cdot C\end{pmatrix} = e^{-x}\begin{pmatrix}-(Ax+B)+A\\-(Cx+D)+C\end{pmatrix}$$

$$e^{-x}\begin{pmatrix}-(Ax+B)+A\\-(Cx+D)+C\end{pmatrix} = \begin{pmatrix}1&4\\1&1\end{pmatrix}\cdot e^{-x}\begin{pmatrix}Ax+B\\Cx+D\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix}e^{-x} \quad /:e^{-x}$$

$$\begin{pmatrix}-Ax-B+A\\-Cx-D+C\end{pmatrix} = \begin{pmatrix}1&4\\1&1\end{pmatrix}\cdot\begin{pmatrix}Ax+B\\Cx+D\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix}$$

$$\begin{pmatrix}-Ax-B+A\\-Cx-D+C\end{pmatrix} = \begin{pmatrix}Ax+B+4Cx+4D+1\\Ax+B+Cx+D+1\end{pmatrix}$$

1) $-Ax-B+A = Ax+B+4Cx+4D+1$
2) $-Cx-D+C = Ax+B+Cx+D+1$

1) $x^1: -A = A+4C$
$x^0: -B+A = B+4D+1$

2) $x^1: -C = A+C$
$x^0: -D+C = B+D+1$

$\cancel{0 = 2A + 4C}$ ... 2násobek rce → skrátíme

$-1 = 2B - A + 4D$

$0 = A + 2C$

$-1 = B - C + 2D$

$-1 = 2B - A + 4D$

$0 = A + 2C \Rightarrow A = -2C$

$-1 = B - C + 2D$

$-1 = 2B + 2C + 4D$

$-1 = B - C + 2D \Rightarrow B = -1 + C - 2D$

$-1 = -2 + 2C - \cancel{4D} + 2C + \cancel{4D}$

$1 = +4C \Rightarrow \boxed{C = \frac{1}{4}}, \boxed{A} = -2 \cdot \frac{1}{4} = \boxed{-\frac{1}{2}}, B = -1 + \frac{1}{4} - 2D =$

$= -\frac{3}{4} - 2D$

$\boxed{D = t}, t \in \mathbb{R}$

$\Downarrow$

$\boxed{B = -\frac{3}{4} - 2t}$

$$\Rightarrow \vec{y}_p = e^{-x} \begin{pmatrix} -\frac{1}{2}x - \frac{3}{4} - 2t \\ \frac{1}{4}x + t \end{pmatrix}, t \in \mathbb{R}$$

volíme $t = 0$: $\vec{y}_p = e^{-x} \begin{pmatrix} -\frac{x}{2} - \frac{3}{4} \\ \frac{x}{4} \end{pmatrix}$

$$\underline{\underline{\vec{y} = c_1 e^{-x} \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 e^{3x} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + e^{-x} \begin{pmatrix} -\frac{x}{2} - \frac{3}{4} \\ \frac{x}{4} \end{pmatrix}}}, c_1, c_2 \in \mathbb{R}$$