# UŽITÍ DVOJNÉHO INTEGRÁLU
# (OBSAH, HMOTNOST, TĚŽIŠTĚ)

## OBSAH MNOŽINY BODŮ V $\mathbb{R}^2$

$$S_A = \iint_A dx\,dy$$

## HMOTNOST ROVINNÉ DESKY

$h(x,y)$ ... hustota v bodě $[x,y]$

$$m = \iint_A h(x,y)\,dx\,dy$$

## SOUŘADNICE TĚŽIŠTĚ ROVINNÉ DESKY

$T = [x_T, y_T]$

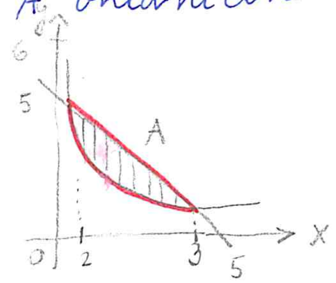
$$x_T = \frac{1}{m} \cdot S_y = \frac{1}{m} \cdot \iint_A x \cdot h(x,y)\,dx\,dy$$

$$y_T = \frac{1}{m} \cdot S_x = \frac{1}{m} \iint_A y \cdot h(x,y)\,dx\,dy$$

## PŘÍKLAD

Vypočítejte obsah množiny $A$, je-li:

a) $A$ ohraničená křivkami $x+y=5$, $xy=6$




$y = 5-x$

$y = \frac{6}{x}$

$A: \quad ? \le x \le ?$

$\frac{6}{x} \le y \le 5-x$

$5 - x = \frac{6}{x} \quad / \cdot x$

$5x - x^2 - 6 = 0$

$x^2 - 5x + 6 = 0$

$(x-2)(x-3) = 0 \Rightarrow 2 \le x \le 3$

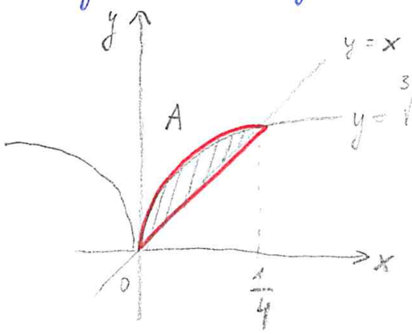
$$S = \int_2^3 \int_{\frac{6}{x}}^{5-x} dy\,dx = \int_2^3 [y]_{\frac{6}{x}}^{5-x} dx = \int_2^3 \left(5 - x - \frac{6}{x}\right) dx = \left[5x - \frac{x^2}{2} - 6\ln x\right]_2^3 =$$

$$= \left(15 - \frac{9}{2} - 6\ln 3\right) - \left(10 - 2 - 6\ln 2\right) = 5 - \frac{9}{2} + 2 - 6\ln 3 + 6\ln 2 =$$

$$= \frac{5}{2} + 6(\ln 2 - \ln 3) = \frac{5}{2} + 6\ln\frac{2}{3}$$

b) A ohraničená krivkami $4y^3 = x^2$, $y = x$

$$4y^3 = x^2 \Rightarrow y = \sqrt[3]{\frac{x^2}{4}}$$




$$x = \sqrt[3]{\frac{x^2}{4}}$$
$$x^3 = \frac{x^2}{4} \quad | \cdot 4$$
$$4x^3 - x^2 = 0$$
$$x^2(4x-1) = 0$$
$$x = 0,\ x = \frac{1}{4}$$

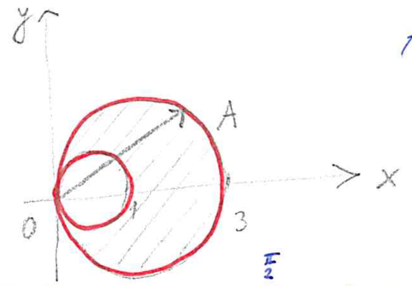
$A: \ 0 \le x \le \frac{1}{4}$
$\quad x \le y \le \sqrt[3]{\frac{x^2}{4}}$

$$S = \int_0^{\frac{1}{4}} \int_x^{\sqrt[3]{\frac{x^2}{4}}} dy\, dx = \int_0^{\frac{1}{4}} \left[ y \right]_x^{\sqrt[3]{\frac{x^2}{4}}} dx = \int_0^{\frac{1}{4}} \left( \sqrt[3]{\frac{x^2}{4}} - x \right) dx = \left[ \frac{1}{\sqrt[3]{4}} \cdot \frac{x^{\frac{5}{3}}}{\frac{5}{3}} - \frac{x^2}{2} \right]_0^{\frac{1}{4}} =$$

$$= \left[ \frac{3}{5\sqrt[3]{4}}\, x \sqrt[3]{x^2} - \frac{x^2}{2} \right]_0^{\frac{1}{4}} = \frac{3}{5\sqrt[3]{4}} \cdot \frac{1}{4} \cdot \sqrt[3]{\frac{1}{16}} - \frac{\frac{1}{16}}{2} = \frac{3}{5\sqrt[3]{4}} \cdot \frac{1}{4} \cdot \frac{1}{2\sqrt[3]{2}} - \frac{1}{32} =$$

$$= \frac{3}{80} - \frac{1}{32} = \frac{1}{160}$$

c) A vnútri kružnice $\rho = 3\cos\varphi$ a vne kružnice $\rho = \cos\varphi$ (popis v polárnych súradniciach)



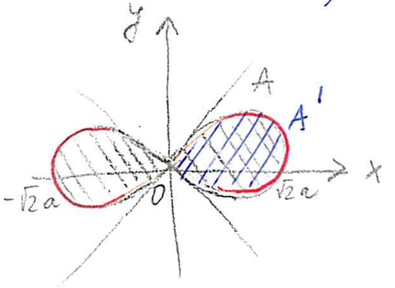

$A: \ -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$
$\quad \cos\varphi \le \rho \le 3\cos\varphi$

$$S = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\cos\varphi}^{3\cos\varphi} \rho\, d\rho\, d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[ \frac{\rho^2}{2} \right]_{\cos\varphi}^{3\cos\varphi} d\varphi = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (9\cos^2\varphi - \cos^2\varphi)\, d\varphi =$$

$$= 4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\varphi\, d\varphi = 4\left[ \frac{\varphi}{2} + \frac{1}{4}\sin 2\varphi \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 4\left[ \left( \frac{\pi}{4} + \frac{1}{4}\cdot 0 \right) - \left( -\frac{\pi}{4} + 0 \right) \right] = 4 \cdot 2 \cdot \frac{\pi}{4} =$$

$$= 2\pi$$

d) A ohraničená lemniskátou $\rho^2 = 2a^2\cos 2\varphi$, $a > 0$ (popis v pol. súradniciach)




$A': \ -\frac{\pi}{4} \le \varphi \le \frac{\pi}{4}$
$\quad 0 \le \rho \le \sqrt{2a^2\cos 2\varphi}$

$$S_A = 2S_{A'} = 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\sqrt{2a^2\cos 2\varphi}} \rho\, d\rho\, d\varphi = 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[ \frac{\rho^2}{2} \right]_0^{\sqrt{2a^2\cos 2\varphi}} d\varphi = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2a^2 \cos 2\varphi\, d\varphi$$

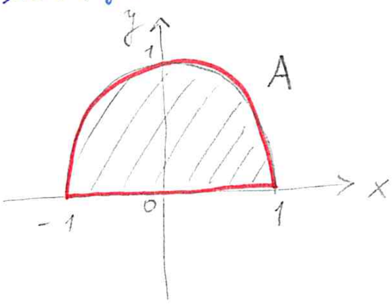
$$= 2a^2\left[ \frac{1}{2}\sin 2\varphi \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = a^2\left(\sin\frac{\pi}{2} + \sin\frac{\pi}{2}\right) = 2a^2$$

(2)

PRIKLAD

Vypočítejte hmotnost a určete souřadnice těžiště rovinné desky $A$ s hustotou $h(x,y)$, je-li:

a) $A$ ohraničená osou $x$ a horní polovinou kružnice $x^2+y^2=1$,
$h(x,y) = x^2+y^2$



$$A: \quad 0 \le \varphi \le \pi, \qquad 0 \le \rho \le 1$$

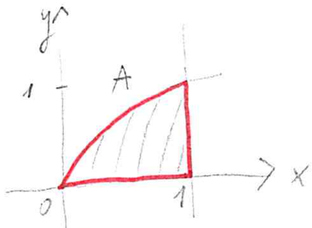
$$m = \int_0^{\pi}\int_0^{1} (\rho^2\cos^2\varphi + \rho^2\sin^2\varphi)\,\rho \, d\rho\, d\varphi = [\varphi]_0^{\pi}\cdot\left[\frac{\rho^4}{4}\right]_0^1 = \pi\cdot\frac{1}{4} = \frac{\pi}{4}$$

$$x_T = \frac{1}{m}\int_0^{\pi}\int_0^{1} \rho\cdot\cos\varphi\cdot\rho^2\cdot\rho\, d\rho\, d\varphi = \frac{4}{\pi}\int_0^{\pi}\cos\varphi\cdot\left[\frac{\rho^5}{5}\right]_0^1 d\varphi = \frac{4}{5\pi}\cdot[\sin\varphi]_0^{\pi} = 0$$

$$y_T = \frac{1}{m}\int_0^{\pi}\int_0^{1} \rho\cdot\sin\varphi\cdot\rho^2\cdot\rho\, d\rho\, d\varphi = \frac{4}{\pi}\int_0^{\pi}\sin\varphi\cdot\left[\frac{\rho^5}{5}\right]_0^1 d\varphi = \frac{4}{5\pi}\cdot[-\cos\varphi]_0^{\pi} = \frac{4}{5\pi}(1+1) = \frac{8}{5\pi}$$

$$T = \left[0, \frac{8}{5\pi}\right]$$

b) $A$ ohraničená osou $x$, přímkou $x=1$ a křivkou $y=\sqrt{x}$, $h(x,y)=x+y$



$$A: \quad 0 \le x \le 1, \qquad 0 \le y \le \sqrt{x}$$

$$m = \int_0^1\int_0^{\sqrt{x}} (x+y)\, dy\, dx = \int_0^1 \left[xy + \frac{y^2}{2}\right]_0^{\sqrt{x}} dx = \int_0^1 \left(x\sqrt{x} + \frac{x}{2}\right) dx = \left[\frac{x^{5/2}}{\frac{5}{2}} + \frac{x^2}{4}\right]_0^1 = \frac{2}{5} + \frac{1}{4} = \frac{8+5}{20} = \frac{13}{20}$$

$$x_T = \frac{1}{m}\int_0^1\int_0^{\sqrt{x}} x(x+y)\, dy\, dx = \frac{20}{13}\int_0^1 \left[x^2y + x\frac{y^2}{2}\right]_0^{\sqrt{x}} dx = \frac{20}{13}\int_0^1 \left(x^{5/2} + \frac{1}{2}x^2\right) dx =$$

$$= \frac{20}{13}\left[\frac{x^{\frac{7}{2}}}{\frac{7}{2}} + \frac{1}{2}\frac{x^3}{3}\right]_0^1 = \frac{20}{13}\left(\frac{2}{7} + \frac{1}{6}\right) = \frac{20}{13}\cdot\frac{19}{42} = \frac{10}{13}\cdot\frac{19}{21} = \frac{190}{273}$$

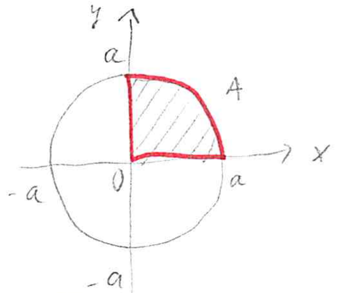
$$y_T = \frac{1}{m}\int_0 \int_0 y(x+y)\,dy\,dx = \frac{20}{13}\int_0 \left[x\frac{y^2}{2}+\frac{y^3}{3}\right]_0 dx = \frac{20}{13}\int_0 \left(\frac{x^2}{2}+\frac{x\sqrt{x}}{3}\right)dx =$$

$$= \frac{20}{13}\left[\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1}{3}\cdot\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^1 = \frac{20}{13}\left(\frac{1}{6}+\frac{2}{15}\right) = \frac{20}{13}\cdot\frac{5+4}{30} = \frac{20\cdot 9}{13\cdot 30} = \frac{2\cdot 9}{13\cdot 3} = \frac{2\cdot 3}{13} =$$

$$= \frac{6}{13}$$

$$T = \left[\frac{190}{273}, \frac{6}{13}\right]$$

c) $A$ ohraničená kružnicí $x^2+y^2=a^2$, $a>0$, souřadnicovými osami, ležící v 1. kvadrantu, $h(x,y)=xy$




$$A: \; 0 \le \varphi \le \frac{\pi}{2}$$
$$0 \le \rho \le a$$

$$m = \int_0^{\frac{\pi}{2}}\int_0^a \rho\cos\varphi\cdot\rho\sin\varphi\cdot\rho\,d\rho\,d\varphi = \left[\frac{\rho^4}{4}\right]_0^a\cdot\int_0^{\frac{\pi}{2}}\sin\varphi\cdot\cos\varphi\,d\varphi = \left|\begin{array}{l} t=\sin\varphi \\ dt=\cos\varphi\,d\varphi \\ 0\to 0 \\ \frac{\pi}{2}\to 1\end{array}\right|$$

$$= \frac{a^4}{4}\cdot\int_0^1 t\,dt = \frac{a^4}{4}\cdot\left[\frac{t^2}{2}\right]_0^1 = \frac{a^4}{8}$$

$$x_T = \frac{1}{m}\int_0^{\frac{\pi}{2}}\int_0^a \rho\cdot\cos\varphi\cdot\rho^2\cos\varphi\cdot\sin\varphi\cdot\rho\,d\rho\,d\varphi = \frac{8}{a^4}\cdot\int_0^{\frac{\pi}{2}}\int_0^a \rho^4\cdot\cos^2\varphi\sin\varphi\,d\rho\,d\varphi$$

$$= \frac{8}{a^4}\left[\frac{\rho^5}{5}\right]_0^a\cdot\int_0^{\frac{\pi}{2}}\cos^2\varphi\cdot\sin\varphi\,d\varphi = \frac{8a^5}{5a^4}\cdot\int_0^{\frac{\pi}{2}}\cos^2\varphi\cdot\sin\varphi\,d\varphi =$$

$$= \left|\begin{array}{l} t=\cos\varphi \\ dt=-\sin\varphi\,d\varphi \\ 0\to 1 \\ \frac{\pi}{2}\to 0\end{array}\right| = \frac{8a}{5}\cdot\left(-\int_1^0 t^2\,dt\right) = \frac{8a}{5}\int_0^1 t^2\,dt = \frac{8a}{5}\left[\frac{t^3}{3}\right]_0^1 =$$

$$= \frac{8a}{15}$$

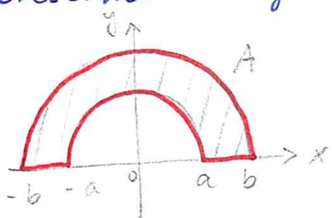
$$y_T = \frac{1}{m}\int_0^{\frac{\pi}{2}}\int_0^a \rho\cdot\sin\varphi\cdot\rho\cos\varphi\cdot\rho\sin\varphi\cdot\rho\,d\rho\,d\varphi = \frac{8}{a^4}\cdot\int_0^{\frac{\pi}{2}}\int_0^a \rho^4\cdot\sin^2\varphi\cos\varphi\,d\rho\,d\varphi$$

$$= \frac{8}{a^4}\left[\frac{\rho^5}{5}\right]_0^a\cdot\int_0^{\frac{\pi}{2}}\sin^2\varphi\cdot\cos\varphi\,d\varphi = \frac{8a^5}{5a^4}\cdot\int_0^1 t^2\,dt = \frac{8a}{5}\cdot\left[\frac{t^3}{3}\right]_0^1 = \frac{8a}{15}$$

$$T = \left[\frac{8a}{15}, \frac{8a}{15}\right]$$

d) $A$ nad osou $x$ a mezi kružnicemi $x^2+y^2=a^2$, $x^2+y^2=b^2$, $0<a<b$

$h(x,y) = x^2$



$$m = \int_0 \int_a (\rho \cos\varphi)^2 \cdot \rho \, d\rho \, d\varphi = \left[\frac{\rho^4}{4}\right]_a \cdot \left[\frac{\varphi}{2} + \frac{1}{2}\sin\varphi \cdot \cos\varphi\right]_0^{\pi} =$$

$$= \frac{b^4 - a^4}{4} \cdot \frac{\pi}{2} = \frac{\pi(b^4 - a^4)}{8}$$

$$x_T = \frac{1}{m}\int_0^{\pi}\int_a^b \rho\cos\varphi \cdot (\rho\cos\varphi)^2 \cdot \rho \, d\rho \, d\varphi = \frac{8}{\pi(b^4-a^4)}\left[\frac{\rho^5}{5}\right]_a^b \cdot \int_0^{\pi} \cos^3\varphi \, d\varphi =$$

$$= \frac{8(b^5-a^5)}{5\pi(b^4-a^4)} \cdot \left[\frac{3}{4}\sin\varphi + \frac{1}{12}\sin 3\varphi\right]_0^{\pi} = 0$$

$$y_T = \frac{1}{m}\int_0^{\pi}\int_a^b \rho\sin\varphi \cdot (\rho\cos\varphi)^2 \cdot \rho \, d\rho \, d\varphi = \frac{-8}{\pi(b^4-a^4)}\left[\frac{\rho^5}{5}\right]_a^b \cdot \int_0^{\pi} -\sin\varphi \cdot \cos^2\varphi \, d\varphi =$$

$$= \left| \begin{array}{l} t = \cos\varphi \\ dt = -\sin\varphi \, d\varphi \\ 0 \to 1 \\ \pi \to -1 \end{array} \right| = \frac{-8(b^5-a^5)}{5\pi(b^4-a^4)} \cdot \int_1^{-1} t^2 \, dt = \frac{8(b^5-a^5)}{5\pi(b^4-a^4)} \cdot \int_{-1}^{1} t^2 \, dt =$$

$$= \frac{8(b^5-a^5)}{5\pi(b^4-a^4)} \cdot \left[\frac{t^3}{3}\right]_{-1}^{1} = \frac{8(b^5-a^5)}{15\pi(b^4-a^4)} \cdot (1+1) = \frac{16(b^5-a^5)}{15\pi(b^4-a^4)}$$

$$(b^5 - a^5) : (b-a) = b^4 + ab^3 + a^2b^2 + a^3b + a^4$$

$$\begin{array}{l}
-b^5 + ab^4 \\
\hline
\quad ab^4 - a^5 \\
\quad -ab^4 + a^2b^3 \\
\hline
\qquad a^2b^3 - a^5 \\
\qquad -a^2b^3 + a^3b^2 \\
\hline
\qquad\quad a^3b^2 - a^5 \\
\qquad\quad -a^3b^2 + a^4b \\
\hline
\qquad\qquad a^4b - a^5 \\
\qquad\qquad -a^4b + a^5 \\
\hline
\qquad\qquad\quad 0
\end{array}$$

$$T = \left[0, \; \frac{16(b^4 + ab^3 + a^2b^2 + a^3b + a^4)}{15\pi(b+a)(b^2+a^2)}\right]$$