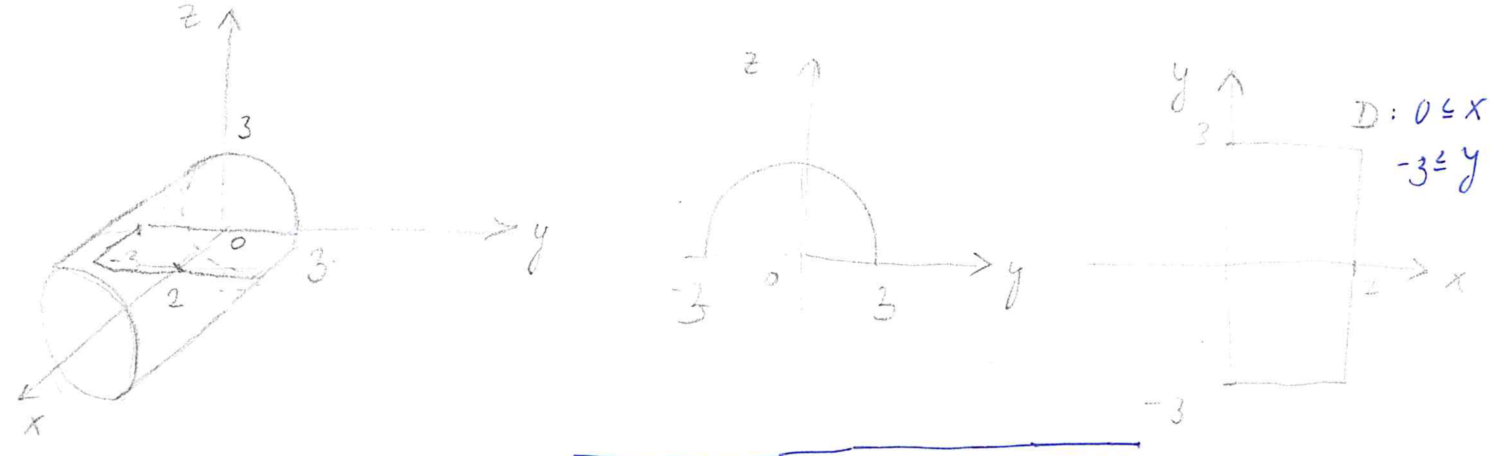
PLOŠNÝ INTEGRÁL 1. DRUHU

PŘÍKLAD 1

$|z| = \sqrt{9-y^2} \Rightarrow (+)$

1b) nalezněte obsah části válcové plochy $y^2+z^2=9$ nad obdélníkem $R = \{[x,y] \in \mathbb{R}^2;\ 0 \le x \le 2,\ -3 \le y \le 3\}$



$D: 0 \le x \le 2$
$-3 \le y \le 3$

$$\text{obsah} = \iint_S 1\,dS = \iint_D 1\cdot\sqrt{1+\left(\frac{\partial}{\partial x}\sqrt{9-y^2}\right)^2+\left(\frac{\partial}{\partial y}\sqrt{9-y^2}\right)^2}\,dx\,dy =$$

$$= \iint_D \sqrt{1+\left[\tfrac{1}{2}\cdot(9-y^2)^{-1/2}\cdot(-2y)\right]^2}\,dx\,dy = \iint_D \sqrt{1+y^2\cdot(9-y^2)^{-1}}\,dx\,dy =$$

$$= \iint_D \sqrt{1+\frac{y^2}{9-y^2}}\,dx\,dy = \int_0^2\left(\int_{-3}^{3}\sqrt{1+\frac{y^2}{9-y^2}}\,dy\right)dx = 2\cdot\int_{-3}^{3}\sqrt{1+\frac{y^2}{9-y^2}}\,dy =$$

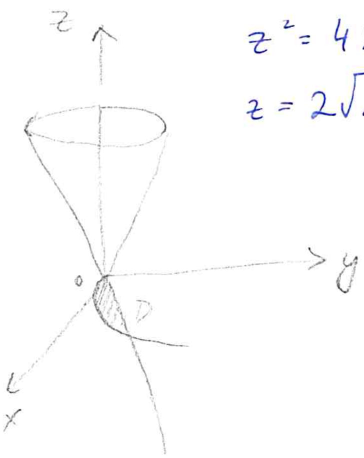
$$= 2\int_{-3}^{3}\sqrt{\frac{9-y^2+y^2}{9-y^2}}\,dy = 2\int_{-3}^{3}\sqrt{\frac{9}{9-y^2}}\,dy = 2\int_{-3}^{3}\sqrt{\frac{9}{9\left(1-\left(\frac{y}{3}\right)^2\right)}}\,dy =$$

$$= 2\int_{-3}^{3}\sqrt{\frac{1}{1-\left(\frac{y}{3}\right)^2}}\,dy = \left|\begin{array}{l} t=\frac{y}{3} \\ dt=\frac{1}{3}\,dy \\ dy = 3\,dt \\ -3 \to -1 \\ 3 \to 1 \end{array}\right| = 2\int_{-1}^{1}\sqrt{\frac{1}{1-t^2}}\cdot 3\,dt =$$

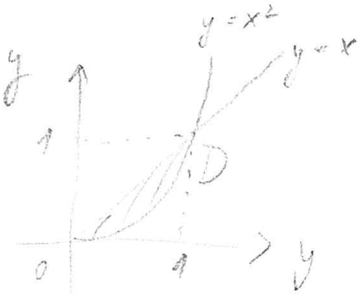
$$= 6\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}\,dt = 6\left[\arcsin t\right]_{-1}^{1} = 6(\arcsin 1 - \arcsin(-1)) =$$

$$= 6\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right) = \underline{6\pi}$$

a) nalezněte obsah části kužele $z^2 = 4x^2+4y^2$, který leží nad oblastí v I. kvadrantu omezenou přímkou $y=x$ a parabolou $y=x^2$.



$$z^2 = 4x^2 + 4y^2$$
$$z = 2\sqrt{x^2+y^2}$$



$$D:\ 0 \le x \le 1$$
$$x^2 \le y \le x$$

$$\text{obsah} = \iint_S 1\, dS = \iint_D 1 \cdot \sqrt{1 + \left(\frac{\partial}{\partial x} 2\sqrt{x^2+y^2}\right)^2 + \left(\frac{\partial}{\partial y} 2\sqrt{x^2+y^2}\right)^2}\, dx\, dy =$$

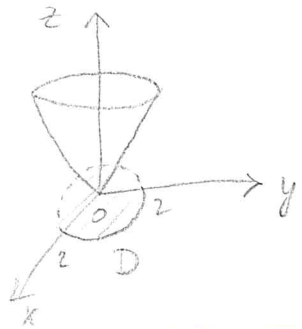
$$= \iint_D \sqrt{1 + \left[2 \cdot \tfrac{1}{2} \cdot (x^2+y^2)^{-1/2} \cdot 2x\right]^2 + \left[2 \cdot \tfrac{1}{2} \cdot (x^2+y^2)^{-1/2} \cdot 2y\right]^2}\, dx\, dy =$$

$$= \iint_D \sqrt{1 + \frac{4x^2}{x^2+y^2} + \frac{4y^2}{x^2+y^2}}\, dx\, dy = \iint_D \sqrt{\frac{5(x^2+y^2)}{x^2+y^2}}\, dx\, dy =$$

$$= \sqrt{5} \iint_D 1\, dx\, dy = \sqrt{5} \int_0^1 \left(\int_{x^2}^{x} dy \right) dx = \sqrt{5} \int_0^1 (x - x^2)\, dx =$$

$$= \sqrt{5} \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \underline{\underline{\frac{\sqrt{5}}{6}}}$$

c) Vypočítejte hmotnost plochy $z = \sqrt{x^2+y^2}$ nad mn. $A = \{[x,y] \in \mathbb{R}^2 \,;\, x^2+y^2 \le 4\}$, je-li hustota v každém bodě rovna vzdálenosti bodu od počátku souřadnic.



$$h(x,y,z) = \sqrt{x^2+y^2+z^2} = \sqrt{x^2+y^2+x^2+y^2} =$$
$$= \sqrt{2} \cdot \sqrt{x^2+y^2}$$

$$D:\ 0 \le \rho \le 2$$
$$0 \le \varphi \le 2\pi$$

$$m = \iint_D \sqrt{2} \cdot \sqrt{x^2+y^2} \cdot \sqrt{1 + \left(\frac{\partial}{\partial x}\sqrt{x^2+y^2}\right)^2 + \left(\frac{\partial}{\partial y}\sqrt{x^2+y^2}\right)^2}\, dx\, dy =$$

$$= \sqrt{2} \iint_D \sqrt{x^2+y^2} \cdot \sqrt{1 + \left[\tfrac{1}{2} \cdot (x^2+y^2)^{-1/2} \cdot 2x\right]^2 + \left[\tfrac{1}{2} \cdot (x^2+y^2)^{-1/2} \cdot 2y\right]^2}$$

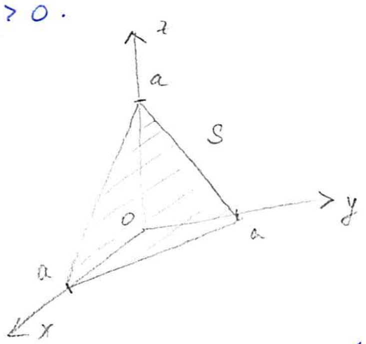
$$dx\, dy = \sqrt{2} \iint_D \sqrt{x^2+y^2} \cdot \sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}}\, dx\, dy$$

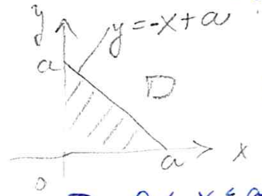
$$= \sqrt{2}\iint_D \sqrt{x^2+y^2}\cdot\sqrt{\frac{2(x^2+y^2)}{x^2+y^2}}\,dx\,dy = \sqrt{2}\cdot\iint_D \sqrt{2}\sqrt{x^2+y^2}\,dx\,dy =$$

$$= 2\iint_D \sqrt{x^2+y^2}\,dx\,dy = 2\int_0^{2\pi}\left(\int_0^{2} \rho\cdot\rho\,d\rho\right)d\varphi = 2\left[\varphi\right]_0^{2\pi}\cdot\left[\frac{\rho^3}{3}\right]_0^2 =$$

$$= 2\cdot 2\pi\cdot\frac{8}{3} = \underline{\underline{\frac{32}{3}\pi}}$$

a) najděte ~~obsah a~~ hm. plochy trojúhelníku o vrcholech $[a,0,0]$, $[0,a,0]$, $[0,0,a]$, $a>0$, a hustotě $h(x,y,z)=k(x+y)$, $k>0$.





$D:\ 0 \le x \le a$
$\quad\ 0 \le y \le a-x$

$bx + cy + dz + e = 0$
$ba + e = 0$
$e = -ab$
$ax + ay + az - a^2 = 0$
$a(x+y+z-a) = 0$
$z = a - x - y$

$$\text{obsah} = \iint_S 1\,dS = \iint_D \sqrt{1+\left(\frac{\partial}{\partial x}(a-x-y)\right)^2+\left(\frac{\partial}{\partial y}(a-x-y)\right)^2}\,dx\,dy =$$

$$= \iint_D \sqrt{1+1+1}\,dx\,dy = \sqrt{3}\iint_D 1\,dx\,dy = \sqrt{3}\int_0^a\left(\int_0^{a-x} 1\,dy\right)dx =$$

$$= \sqrt{3}\int_0^a (a-x)\,dx = \sqrt{3}\left[ax-\frac{x^2}{2}\right]_0^a = \sqrt{3}\left(a^2-\frac{a^2}{2}\right) = \underline{\underline{\frac{\sqrt{3}}{2}a}}$$

$$m = \iint_S h(x,y)\,dS = k\iint_D (x+y)\sqrt{3}\,dx\,dy = k\sqrt{3}\int_0^a\left(\int_0^{a-x}(x+y)\,dy\right)dx =$$

$$= k\sqrt{3}\int_0^a\left[xy+\frac{y^2}{2}\right]_0^{a-x}dx = k\sqrt{3}\int_0^a\left[x(a-x)+\frac{1}{2}(a-x)^2\right]dx =$$

$$= k\sqrt{3}\int_0^a\left(ax-x^2+\frac{1}{2}a^2-ax+\frac{1}{2}x^2\right)dx = k\sqrt{3}\int_0^a\left(-\frac{1}{2}x^2+\frac{1}{2}a^2\right)dx =$$

$$= k\sqrt{3}\left[-\frac{1}{2}\frac{x^3}{3}+\frac{1}{2}a^2x\right]_0^a = k\sqrt{3}\left(-\frac{1}{6}a^3+\frac{1}{2}a^3\right) = k\sqrt{3}\cdot\frac{1}{3}a^3 =$$

$$= \underline{\underline{\frac{\sqrt{3}\cdot k a^3}{3}}}$$

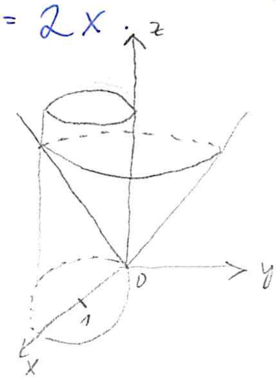
2b.) najdite obsah části kuželové plochy $z=\sqrt{x^2+y^2}$ ležící uvnitř válce $x^2+y^2=2x$.

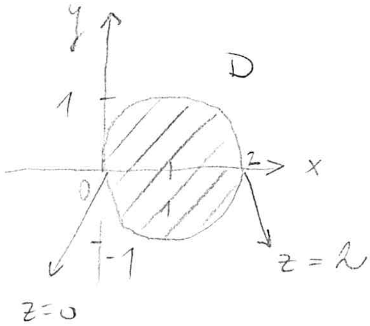
$x^2+y^2-2x=0$

$(x^2-2x)+y^2=0$

$(x^2-2x+1)-1+y^2=0$

$(x-1)^2+y^2=1$





$$\text{obsah} = \iint_S 1\,ds = \iint_D 1\sqrt{1+\left(\frac{\partial}{\partial x}\sqrt{x^2+y^2}\right)^2+\left(\frac{\partial}{\partial y}\sqrt{x^2+y^2}\right)^2}\,dxdy =$$

$$= \iint_D \sqrt{1+\left(\frac{1}{\cancel{2}}(x^2+y^2)^{-1/2}\cdot \cancel{2}x\right)^2+\left(\frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2y\right)^2}\,dxdy =$$

$$= \iint_D \sqrt{1+\frac{x^2+y^2}{x^2+y^2}}\,dxdy = \sqrt{2}\iint_D dxdy = \left| \begin{array}{l} x=u+1 \\ y=v \\ |J|=1 \\ D\to D' \end{array} \right| =$$

$$= \sqrt{2}\iint_{D'} dudv = \left| \begin{array}{l} u=\rho\cos\varphi \\ v=\rho\sin\varphi \\ |J|=\rho \\ D'\to D'' \end{array} \right| = \sqrt{2}\iint_{D''}\rho\,d\rho\,d\varphi = \sqrt{2}\int_0^{2\pi}\left(\int_0^1 \rho\,d\rho\right)d\varphi$$

$$= \sqrt{2}\cdot 2\pi\cdot\frac{1}{2} = \underline{\underline{\sqrt{2}\cdot\pi}}$$

PLOŠNÝ INTEGRÁL 1. DRUHU

$$\iint_S f(x,y,z)\,dS = \iint_A f(x,y,g(x,y))\,dS = \iint_A f(x,y,g(x,y))\cdot\sqrt{1+g_x'^2+g_y'^2}\,dx\,dy.$$

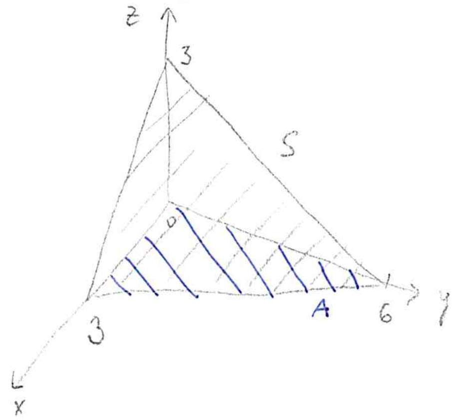
kde S je plocha zadaná funkcí $z=g(x,y)$ a A je její průmět do roviny xy

PŘÍKLAD

Vypočtěte:

a) $\iint_S (y^2+2yz)\,dS$, kde S je část roviny $2x+y+2z-6=0$ ležící v 1. oktantu

$$z=\frac{-2x-y+6}{2} = -x-\frac{y}{2}+3 = g(x,y)$$



$A:\ 0\le x\le 3$

$0\le y\le\ ?\ :$

$0=-x-\frac{y}{2}+3$

$\frac{y}{2}=3-x$

$y=6-2x$

$\Rightarrow 0\le y\le 6-2x$

$$\iint_S (y^2+2yz)\,dS = \iint_A \left[y^2+2y\cdot\left(-x-\frac{y}{2}+3\right)\right]\cdot\sqrt{1+(-1)^2+\left(-\frac{1}{2}\right)^2}\,dx\,dy =$$

$$=\int_0^3\int_0^{6-2x}\left(y^2-2xy-y^2+6y\right)\sqrt{2+\frac{1}{4}}\,dy\,dx = \int_0^3\int_0^{6-2x}(-2xy+6y)\sqrt{\frac{9}{4}}\,dy\,dx =$$

$$=\frac{3}{2}\int_0^3\int_0^{6-2x}(-2xy+6y)\,dy\,dx = \frac{3}{2}\int_0^3\left(-2x\left[\frac{y^2}{2}\right]_0^{6-2x}+6\left[\frac{y^2}{2}\right]_0^{6-2x}\right)dx =$$

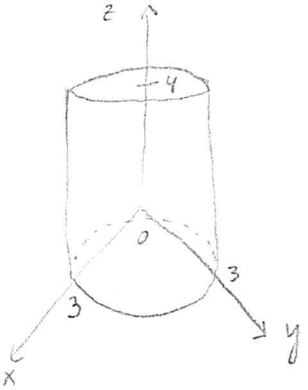
$$=\frac{3}{2}\int_0^3\left[-\frac{2x}{2}(6-2x)^2+3(6-2x)^2\right]dx = \frac{3}{2}\int_0^3\left[(3-x)(36-24x+4x^2)\right]dx =$$

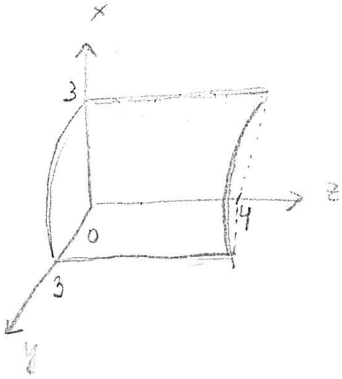
$$=\frac{3}{2}\int_0^3\left(108-72x+12x^2-36x+24x^2-4x^3\right)dx = \frac{3}{2}\int_0^3\left(-4x^3+36x^2-108x+108\right)$$

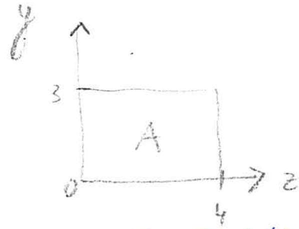
$$dx = \frac{3}{2}\left[-4\frac{x^4}{4}+36\frac{x^3}{3}-108\frac{x^2}{2}+108x\right]_0^3 = \frac{3}{2}(-81+324-486+324) =$$

$$=\frac{243}{2}$$

b) $\iint\limits_S (x+z)\, dS$, kde S je část válcové plochy $x^2+y^2=9$ v I. oktantu odříznutá rovinou $z=4$ $\quad \Rightarrow x=\sqrt{9-y^2}$







$A: \; 0 \le z \le 4$
$\quad\;\; 0 \le y \le 3$

$$\iint\limits_S (x+z)\, dS = \iint\limits_A \left(\sqrt{9-y^2}+z\right)\cdot\sqrt{1+\left[\tfrac{1}{2}\cdot(9-y^2)^{-\frac{1}{2}}\cdot(-2y)\right]^2+0^2}\; dy\, dz =$$

$$= \int_0^3\int_0^4 \left(\sqrt{9-y^2}+z\right)\cdot\sqrt{1+\tfrac{1}{4}\cdot(9-y^2)^{-1}\cdot 4y^2}\; dz\, dy =$$

$$= \int_0^3\int_0^4 \left(\sqrt{9-y^2}+z\right)\sqrt{1+\frac{y^2}{9-y^2}}\; dz\, dy = \int_0^3\int_0^4 \left(\sqrt{9-y^2}+z\right)\cdot\sqrt{\frac{9}{9-y^2}}\; dz\, dy =$$

$$= 3\int_0^3\int_0^4 \left(\sqrt{9-y^2}+z\right)\cdot\frac{1}{\sqrt{9-y^2}}\; dz\, dy = 3\int_0^3\int_0^4 \left(1+\frac{z}{\sqrt{9-y^2}}\right) dz\, dy =$$

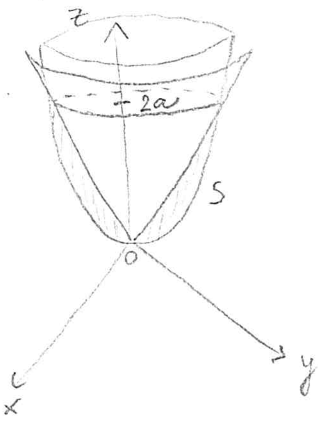
$$= 3\int_0^3 \left(\left[z\right]_0^4 + \frac{1}{\sqrt{9-y^2}}\cdot\left[\frac{z^2}{2}\right]_0^4\right) dy = 3\int_0^3 \left(4+\frac{8}{\sqrt{9-y^2}}\right) dy =$$

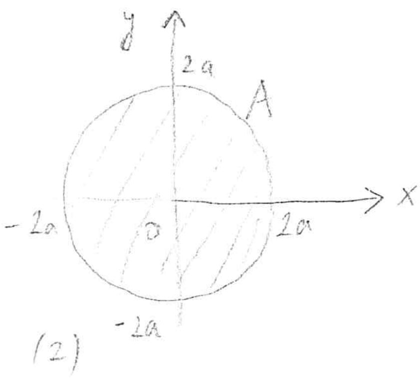
$$= 3\left[4y\right]_0^3 + 24\cdot\int_0^3 \frac{1}{\sqrt{9-y^2}}\, dy = 36 + 24\cdot\frac{\pi}{2} = 36+12\pi$$

c) $\iint\limits_S (a^2+x^2+y^2)\, dS$, kde S je část paraboloidu $x^2+y^2=2az$ odříznutá kuželem $x^2+y^2=z^2$ $\quad \Rightarrow z=\frac{x^2}{2a}+\frac{y^2}{2a}$

$2az = z^2 \quad /:z$

$2a = z$





A :

$$\iint_S (a^2+x^2+y^2)\, dS = \iint_A (a^2+x^2+y^2)\sqrt{1+\left(\frac{2x}{2a}\right)^2+\left(\frac{2y}{2a}\right)^2}\, dx\, dy =$$

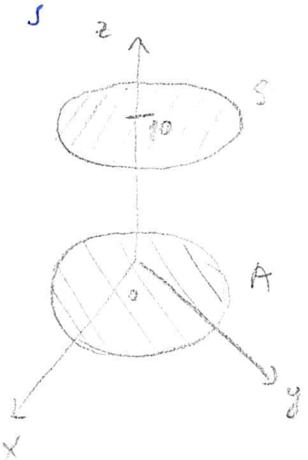
$$= \iint_A (a^2+x^2+y^2)\sqrt{1+\frac{x^2}{a^2}+\frac{y^2}{a^2}}\, dx\, dy = \iint_A (a^2+x^2+y^2)\sqrt{\frac{a^2+x^2+y^2}{a^2}}\, dx\, dy =$$

$$= \iint_A \frac{1}{a}(a^2+x^2+y^2)^{3/2}\, dx\, dy = \int_0^{2\pi}\int_0^{2a} \frac{1}{a}(a^2+\rho^2\cos^2\varphi+\rho^2\sin^2\varphi)\,\rho\, d\rho\, d\varphi =$$

$$= \frac{1}{a}\int_0^{2\pi}\int_0^{2a}(a^2+\rho^2)\rho\, d\rho\, d\varphi = \frac{1}{a}\int_0^{2\pi}\left(a^2\left[\frac{\rho^2}{2}\right]_0^{2a}+\left[\frac{\rho^4}{4}\right]_0^{2a}\right) d\varphi =$$

$$= \frac{1}{a}\int_0^{2\pi}(2a^4+4a^4)\, d\varphi = 6a^3[\varphi]_0^{2\pi} = 12\pi a^3$$

d) $\iint_S (x-2y+z)\, dS$, kde $S = \{[x,y,z] : z=10,\ x^2+y^2\le 1\}$



$A:\ 0\le\rho\le 1$
$\quad\ 0\le\varphi\le 2\pi$

$$\iint_S (x-2y+z)\, dS = \iint_A (x-2y+10)\sqrt{1+0^2+0^2}\, dx\, dy = \iint_A (x-2y+10)\, dx\, dy =$$

$$= \int_0^{2\pi}\int_0^1 (\rho\cos\varphi - \rho\sin\varphi + 10)\rho\, d\rho\, d\varphi = \int_0^{2\pi}\left(\cos\varphi\left[\frac{\rho^3}{3}\right]_0^1 - \sin\varphi\left[\frac{\rho^3}{3}\right]_0^1 + 10\left[\frac{\rho^2}{2}\right]_0^1\right)$$

$$d\varphi = \frac{1}{3}\int_0^{2\pi}(\cos\varphi - \sin\varphi + 5)\, d\varphi = \frac{1}{3}[\sin\varphi + \cos\varphi]_0^{2\pi} + 5[\varphi]_0^{2\pi} = \frac{1}{3}\big((0+1)-(0+1)\big) +$$

$$+ 10\pi = 10\pi$$

PLOŠNÝ INTEGRÁL 2. DRUHU

$a = u \cdot i + v \cdot j + w \cdot k$

$$\iint_S a \cdot n \, dS = \iint_S u \, dy dz + v \, dx dz + w \, dx dy = \iint_S (u \cdot i + v \cdot j + w \cdot k) \cdot$$

$$\cdot \frac{(-g_x', -g_y', 1)}{\sqrt{g_x'^2 + g_y'^2 + 1}} \, dS = \iint_A (-u g_x' - v g_y' + w) \cdot \frac{1}{\sqrt{g_x'^2 + g_y'^2 + 1}} \cdot \sqrt{g_x'^2 + g_y'^2 + 1}$$

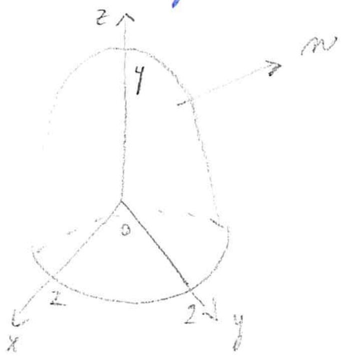
$$dx dy = \iint_A (-u g_x' - v g_y' + w) \, dx dy$$

$$\iint_S a \cdot n_1 \, dS = - \iint_S a \cdot n_2 \, dS$$

PŘÍKLAD

Vypočítejte $\iint_S a \cdot n \, dS$, kde:

a) $a(x,y,z) = x \cdot i + y \cdot j + z \cdot k$, S je část paraboloidu $g(x,y) = z = 4 - x^2 - y^2$ ležící nad rovinou xy orientovaná vektorem normály směřujícím do poloprostoru neobsahující počátek



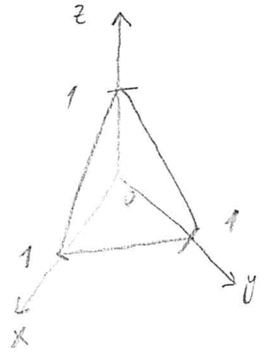
$A: x^2 + y^2 \leq 4$
$0 \leq \rho \leq 2$
$0 \leq \varphi \leq 2\pi$

$$\iint_S a \cdot n \, dS = \iint_A (x, y, 4 - x^2 - y^2)(2x, 2y, 1) \, dx dy = \iint_A (2x^2 + 2y^2 + 4 - x^2 - y^2)$$

$$dx dy = \iint_A (x^2 + y^2 + 4) \, dx dy = \int_0^{2\pi}\int_0^2 (\rho^2 \cos^2\varphi + \rho^2 \sin^2\varphi + 4) \rho \, d\rho \, d\varphi =$$

$$= \int_0^{2\pi}\int_0^2 (4\rho + \rho^3) \, d\rho \, d\varphi = [\varphi]_0^{2\pi} \cdot \left[4\frac{\rho^2}{2} + \frac{\rho^4}{4}\right]_0^2 = 2\pi \cdot (8 + 4) = 24\pi$$

b) $a = 3z \cdot i - 4 \cdot j + y \cdot k$, S je část roviny $x + y + z = 1$ v 1. oktantu orientovaná vektorem normály směřujícím do poloprostoru neobsahující počátek



$z = 1 - x - y \quad \Rightarrow \quad G_1(x,y,z) = -x - y - z + 1$

$\text{grad}\, G_1 = (-1; -1; -1)$

$A: \quad 0 \le x \le 1, \quad 0 \le y \le 1 - x$

$G_2(x,y,z) = (1; 1; 1)$

$\Downarrow$

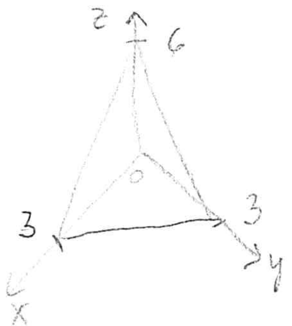
$G(x,y,z) = x + y + z - 1$

$$\iint_S a\cdot n\, dS = \iint_A (3(1-x-y) - 4,\ y)\cdot(1,1,1)\, dx\, dy = \int_0^1\int_0^{1-x} (3 - 3x - 3y - 4 + y)\, dy\, dx$$

$$= \int_0^1\int_0^{1-x} (-3x - 2y - 1)\, dy\, dx = \int_0^1 \left[-3xy - y^2 - y\right]_0^{1-x} dx = \int_0^1 \left[-3x(1-x) - (1-x)^2 - (1-x)\right] dx = \int_0^1 (-3x + 3x^2 - 1 + 2x - x^2 - 1 + x)\, dx =$$

$$= \int_0^1 (2x^2 - 2)\, dx = \left[2\frac{x^3}{3} - 2x\right]_0^1 = \frac{2}{3} - 2 = -\frac{4}{3}$$

c) $a = xy\cdot i - x^2\cdot j + (x+z)\cdot k$, S je část roviny $2x + 2y + z = 6$ v 1. oktantu orientovaná vektorem normály směřujícím do poloprostoru obsahujícího počátek



$z = 6 - 2x - 2y = 2(3 - x - y)$

$y = \frac{6 - 2x - z}{2} = 3 - x - \frac{z}{2}$

$A: \quad 0 \le x \le 3, \quad 0 \le y \le 3 - x$

$$\iint_S a\cdot n\, dS = -\iint_A (xy,\ -x^2,\ x + 2(3-x-y))\cdot(2,2,1)\, dx\, dy =$$

$$= -\iint_A (2xy - 2x^2 + x + 6 - 2x - 2y)\, dx\, dy = -\iint_A (2xy - 2x^2 - x - 2y + 6)\, dx\, dy = -\int_0^3\int_0^{3-x} (2xy - 2x^2 - x - 2y + 6)\, dy\, dx = -\int_0^3 \left[2x\frac{y^2}{2} - 2x^2 y - xy - 2\frac{y^2}{2} + 6y\right]_0^{3-x} dx = -\int_0^3 \left[xy^2 - 2x^2 y - xy - y^2 + 6y\right]_0^{3-x} dx =$$

$$= -\int_0^3 \left[x(3-x)^2 - 2x^2(3-x) - x(3-x) - (3-x)^2 + 6(3-x)\right] dx =$$

$$= -\int_0^3 (9x - 6x^2 + x^3 - 6x^2 + 2x^3 - 3x + \cancel{x^2} - 9 + \cancel{6x} - \cancel{x^2} + 18 - \cancel{6x})\, dx =$$

$$= -\int_0 (3x^3 - 12x^2 + 6x + 9)\,dx = -\left[3\frac{x^4}{4} - 12\frac{x^3}{3} + 6\frac{x^2}{2} + 9x\right]_0 = -\left(3\cdot\frac{81}{4} - 4\cdot 27 + 27 + 27\right) = -\frac{27}{4}$$