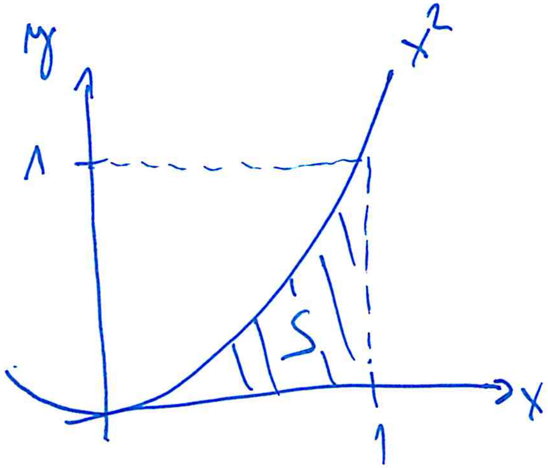
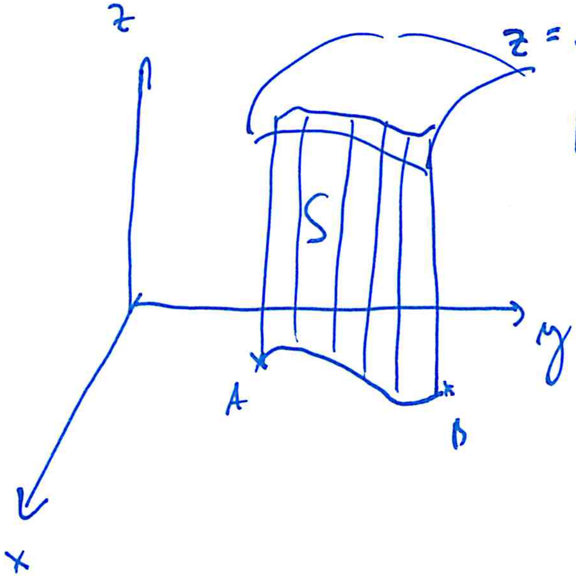
DVOJNÝ INTEGRÁL

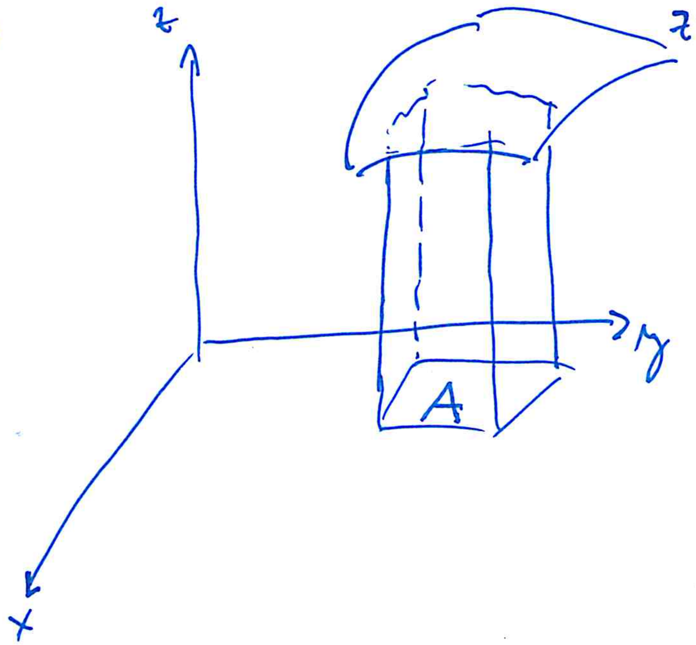
$$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} = S$$





$z = f(x,y)$

křivkový integrál



$z = f(x,y)$

OBJEM MEZI OBRAZCEM A GRAFEM FUNKCE

$$\iint_A f(x,y)\,dx\,dy$$

Věta o dvojnásobném integrálu

obrazec A je obrazec typu

I.

$$a \le x \le b$$
$$d(x) \le y \le h(x)$$

$$\iint_A f(x,y)\,dx\,dy = \int_a^b \left[\int_{d(x)}^{h(x)} f(x,y)\,dy \right] dx$$

Dvojný i. Dvojnásobný i.

II.

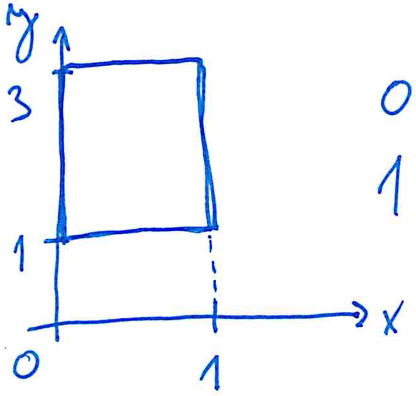
$$c \le y \le d$$
$$l(y) \le x \le p(y)$$

$$\iint_A f(x,y)\,dx\,dy = \int_c^d \left[\int_{l(y)}^{p(y)} f(x,y)\,dx \right] dy$$

vnitřní

vnější

Příklady: $\iint\limits_A x^2 y \, dx\, dy$ A je obdélník $\langle 0,1\rangle \times \langle 1,3\rangle$



$0 \le x \le 1$

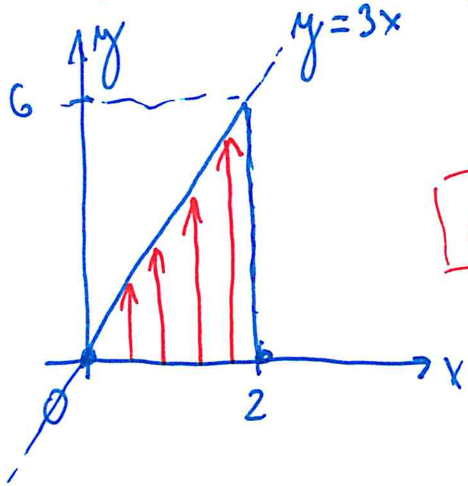
$1 \le y \le 3$

$$\iint\limits_A x^2 y \, dx\, dy = \int_0^1 \left[\int_1^3 x^2 y \, dy \right] dx = \int_0^1 4x^2 \, dx = 4\left[\frac{x^3}{3}\right]_0^1 = \frac{4}{3}$$

$$\int_1^3 x^2 y \, dy = x^2 \left[\frac{y^2}{2}\right]_1^3 = x^2\left(\frac{9}{2} - \frac{1}{2}\right) = 4x^2$$

$$\iint_B 2x\,dx\,dy$$

B je Δ s vrcholy $[0,0]$, $[2,0]$, $[2,6]$

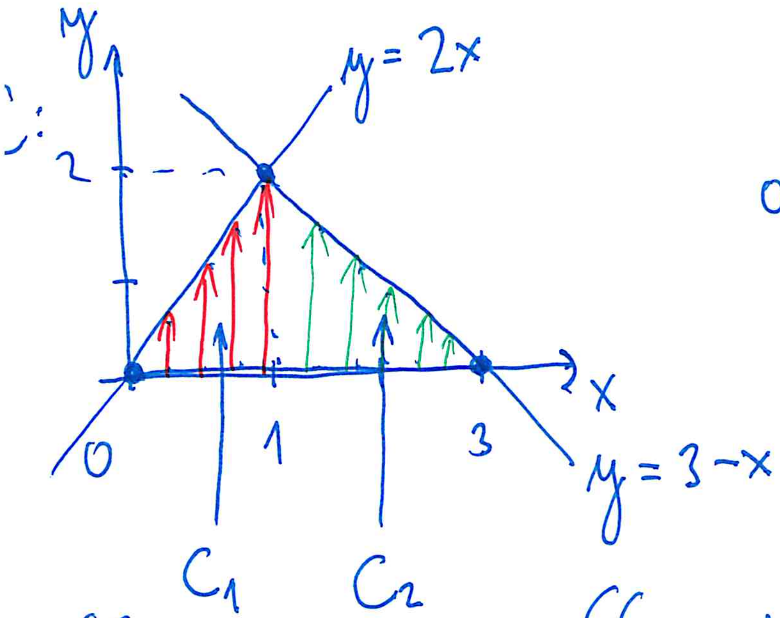


$0 \le x \le 2$

$\boxed{0 \le y \le 3x}$

$$\iint_B 2x\,dx\,dy = \int_0^2 \left[\int_0^{3x} 2x\,dy \right] dx = \int_0^2 6x^2\,dx = \left[2x^3\right]_0^2 = \underline{\underline{16}}$$

$$\int_0^{3x} 2x\,dy = 2x\left[y\right]_0^{3x} = 2x \cdot 3x = 6x^2$$



$$0 \le x \le 1 \qquad \qquad 1 \le x \le 3$$
$$0 \le y \le 2x \quad \text{a} \quad 0 \le y \le 3-x$$

$$\iint_C 1\,dx\,dy = \iint_{C_1} 1\,dx\,dy + \iint_{C_2} 1\,dx\,dy = \int_0^1 \left(\int_0^{2x} 1\,dy \right) dx + \int_1^3 \left(\int_0^{3-x} 1\,dy \right) dx =$$

$$= \int_0^1 2x\,dx + \int_1^3 (3-x)\,dx = \left[x^2\right]_0^1 + \left[3x - \frac{x^2}{2}\right]_1^3 = 1 + 9 - \frac{9}{2} - \left(3 - \frac{1}{2}\right) =$$

$$\int_0^{2x} 1\,dy = \left[y\right]_0^{2x} = 2x \qquad \int_0^{3-x} 1\,dy = \left[y\right]_0^{3-x} = 3 - x$$

$$= 7 - 4 = \underline{\underline{3}}$$

Výpočet dvojného integrálu v polárních souřadnicích

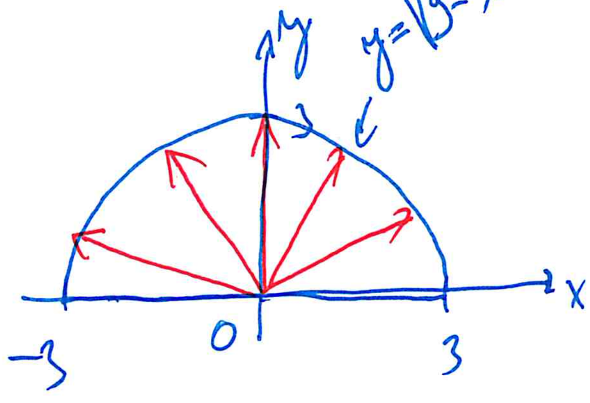
Pomocí substituce $x = \varrho \cos\varphi$ a $y = \varrho \sin\varphi$ dostaneme pro spojitou funkci $z = f(x,y) = f(\varrho\cos\varphi, \varrho\sin\varphi) = g(\varrho,\varphi)$. Platí:

$$\boxed{\iint_A f(x,y)\,dx\,dy = \iint_{\bar{A}} g(\varrho,\varphi)\cdot\varrho\,d\varrho\,d\varphi}$$

$$\iint_D (x^2+y^2)\,dx\,dy \qquad \begin{aligned} x &= \varrho\cos\varphi \\ y &= \varrho\sin\varphi \end{aligned}$$

D je horní půlkruh $S=[0,0]$

$R = 3$



$0 \leq \varphi \leq \pi$

$0 \leq \varrho \leq 3$

$$\iint_D (x^2+y^2)\,dx\,dy = \iint_{\overline{D}} \big(\underbrace{\varrho^2\cos^2\varphi + \varrho^2\sin^2\varphi}_{=\varrho^2}\big)\,\varrho\,d\varrho\,d\varphi = \iint_{\overline{D}} \varrho^3\,d\varrho\,d\varphi = \int_0^{\pi}\Big(\int_0^3 \varrho^3\,d\varrho\Big)\,d\varphi =$$

$$= \int_0^{\pi}\Big[\frac{\varrho^4}{4}\Big]_0^3\,d\varphi = \int_0^{\pi}\frac{81}{4}\,d\varphi = \frac{81}{4}\,[\varphi]_0^{\pi} = \frac{81}{4}\pi$$

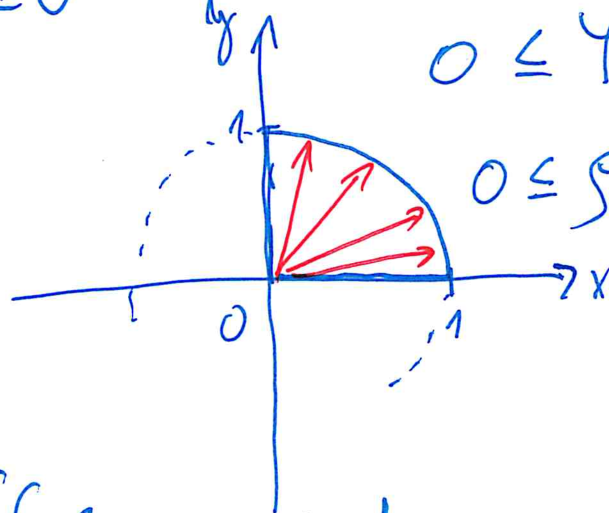
$$\iint_E 2xy\,dx\,dy$$

E: čtvrtkruh $x \geq 0$, $y \geq 0$, $x^2+y^2 \leq 1$

$x = \rho\cos\varphi$

$y = \rho\sin\varphi$

$\sin 2\varphi = 2\sin\varphi\cos\varphi$



$0 \leq \varphi \leq \frac{\pi}{2}$

$0 \leq \rho \leq 1$

$$\iint_E 2xy\,dx\,dy = \iint_E 2\rho\cos\varphi\,\rho\sin\varphi\,\rho\,d\rho\,d\varphi = \iint_E \rho^3\sin 2\varphi\,d\rho\,d\varphi =$$

$$= \int_0^{\frac{\pi}{2}}\left[\int_0^1 \rho^3\sin 2\varphi\,d\rho\right]d\varphi = \int_0^{\frac{\pi}{2}}\sin 2\varphi\left[\frac{\rho^4}{4}\right]_0^1 d\varphi = \frac{1}{4}\int_0^{\frac{\pi}{2}}\sin 2\varphi\,d\varphi =$$

$$= \frac{1}{4}\left[-\cos(2\varphi)\cdot\frac{1}{2}\right]_0^{\frac{\pi}{2}} = \frac{1}{8}(1-(-1)) =$$

$$= \frac{1}{4}$$