Plošný integrál 1. druhu

Je-li S plocha zadaná jako graf $z = f(x,y)$ na uzavřené množině D ($\subset E_2$), definujeme integrál funkce $F = F(x,y,z)$ na ploše S jako:

$$\iint\limits_S F(x,y,z)\,\underline{dS} = \iint\limits_D F(x,y,f(x,y))\underline{\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dx\,dy}.$$

→ $dS = \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dx\,dy$

→ integrujeme přes D - což je průmět plochy S do roviny xy

využití: máme plocha $S: z = f(x,y)$ s plošnou hustotou $h(x,y,z)$

Povrch (obsah) plochy S: $P(S) = \iint_S 1 \, dS$

Hmotnost plochy S: $m(S) = \iint_S h(x,y,z) \, dS$

Souřadnice těžiště plochy S: $T = [x_T, y_T, z_T]$

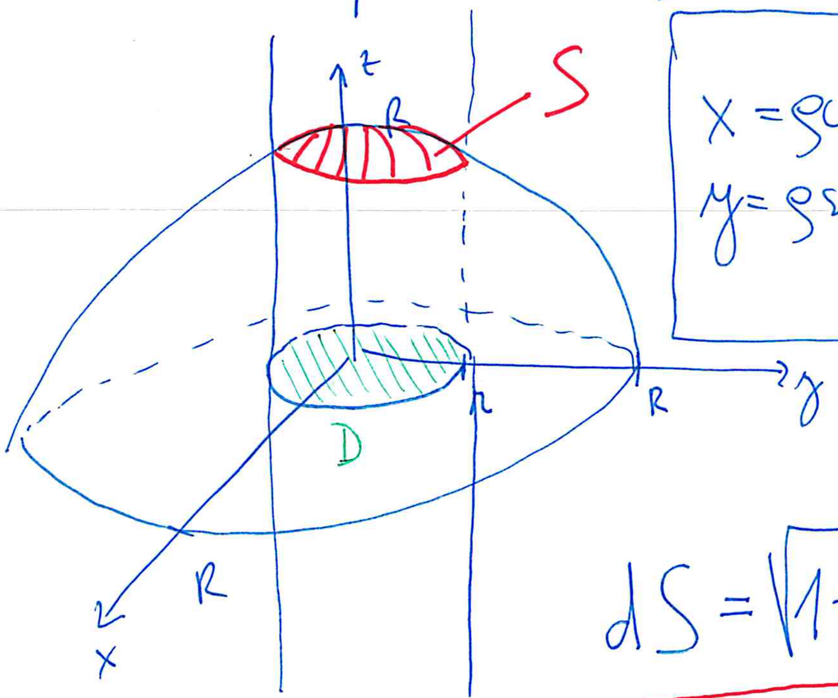
$$x_T = \frac{1}{m} \iint_S x \cdot h(x,y,z) \, dS$$

Momenty setrvačnosti plochy S vzhledem k ose z:

$$I_z = \iint_S \underbrace{(x^2+y^2)}_{R^2} \cdot \underbrace{h(x,y,z) \, dS}_{dm}$$

Příklad: Spočítejte povrch sféry $x^2+y^2+z^2=R^2$ uvnitř válcové plochy $x^2+y^2=r^2$ pro $z\ge 0$, $R\ge r$.

$$\to z=\sqrt{R^2-x^2-y^2}=f(x,y)$$



$$\boxed{\begin{array}{ll} x=\varrho\cos\varphi & 0\le\varrho\le r \\ y=\varrho\sin\varphi & 0\le\varphi\le 2\pi \end{array}}$$

$$\frac{\partial f}{\partial x}=\frac{-x}{\sqrt{R^2-x^2-y^2}}$$

$$\frac{\partial f}{\partial y}=\frac{-y}{\sqrt{R^2-x^2-y^2}}$$

$$\underline{dS=\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dx\,dy}=\sqrt{1+\frac{x^2}{R^2-x^2-y^2}+\frac{y^2}{R^2-x^2-y^2}}\,dx\,dy=$$

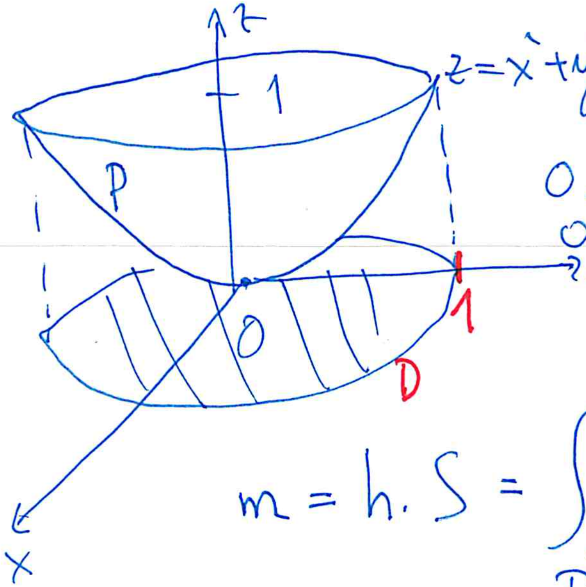
$$=\sqrt{\frac{R^2}{R^2-x^2-y^2}}\,dx\,dy=\frac{R}{\sqrt{R^2-x^2-y^2}}\,dx\,dy$$

$$P=\iint_S 1\,dS=\iint_D\frac{R}{\sqrt{R^2-x^2-y^2}}\,dx\,dy=\iint_D\frac{R}{\sqrt{R^2-\varrho^2}}\,\varrho\,d\varrho\,d\varphi=\left|\begin{array}{l} t=R^2-\varrho^2 \\ dt=-2\varrho\,d\varrho \\ -\frac{1}{2}dt=\varrho\,d\varrho \end{array}\right|=$$

$$=\int_0^{2\pi}\left[\int_0^{r}\frac{R\varrho}{\sqrt{R^2-\varrho^2}}\,d\varrho\right]d\varphi=-\frac{\cancel{2}\pi R}{\cancel{2}}\int_{R^2}^{R^2-r^2}\frac{1}{\sqrt{t}}\,dt=-\pi R\left[\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right]_{R^2}^{R^2-r^2}=\boxed{2\pi R\left(R-\sqrt{R^2-r^2}\right)}$$

Nalezněte souřadnice těžiště části paraboloidu $z = x^2 + y^2$ ohraničeného $z = 1$ s plošnou hustotou $h(x,y,z) = 1$

$x_T = y_T = 0$



$z = x^2 + y^2 = f(x,y)$

$0 \le \varrho \le 1$

$0 \le \varphi \le 2\pi$

$\frac{\partial f}{\partial x} = 2x$

$\frac{\partial f}{\partial y} = 2y$

$$dS = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\,dy = \sqrt{1 + 4x^2 + 4y^2}\, dx\,dy$$

$$m = h \cdot S = \iint_P 1\, dS = \iint_D \sqrt{1 + 4x^2 + 4y^2}\, dx\,dy = \iint_D \sqrt{1 + 4\varrho^2}\, \varrho\, d\varrho\, d\varphi =$$

$x = \varrho \cos\varphi$

$y = \varrho \sin\varphi$

$$= \int_0^{2\pi} \left(\int_0^1 \sqrt{1 + 4\varrho^2} \cdot \varrho\, d\varrho \right) d\varphi = 2\pi \cdot \frac{1}{8} \int_1^5 \sqrt{t}\, dt = \frac{\pi}{4} \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^5 =$$

$t = 1 + 4\varrho^2$

$dt = 8\varrho\, d\varrho$

$\frac{1}{8} dt = \varrho\, d\varrho$

$$= \frac{\pi}{4} \cdot \frac{2}{3} \left(\sqrt{125} - 1 \right) = \underline{\underline{\frac{\pi}{6} \left(\sqrt{125} - 1 \right)}}$$

$$dS = \sqrt{1+4x^2+4y^2}\,dx\,dy, \quad f(x,y) = z = x^2+y^2, \quad h(x,y,z) = 1, \quad m = \frac{\pi}{6}\left(\sqrt{125}-1\right)$$

$$0 \le \rho \le 1$$
$$0 \le \varphi \le 2\pi$$

$$z_T = \frac{1}{m}\iint_P z\,dS = \frac{6}{\pi\left(\sqrt{125}-1\right)} \cdot \frac{\pi}{16}\left(\frac{2}{5}\left(5^{\frac{5}{2}}-1\right) - \frac{2}{3}\left(5^{\frac{3}{2}}-1\right)\right)$$

$$\iint_P z\,dS = \iint_D (x^2+y^2)\sqrt{1+4(x^2+y^2)}\,dx\,dy = \iint \rho^2\sqrt{1+4\rho^2}\cdot\rho\,d\rho\,d\varphi =$$

$$= \int_0^{2\pi}\left(\int_0^1 \rho^2\sqrt{1+4\rho^2}\cdot\rho\,d\rho\right)d\varphi = \frac{2\pi}{4\cdot 8}\int_1^5 (t-1)\sqrt{t}\,dt = \frac{\pi}{16}\int_1^5 \left(t^{\frac{3}{2}} - t^{\frac{1}{2}}\right)dt =$$

$$t = 1+4\rho^2$$
$$dt = 8\rho\,d\rho$$
$$\frac{1}{8}dt = \rho\,d\rho$$
$$\rho^2 = \frac{t-1}{4}$$

$$= \frac{\pi}{16}\left[\frac{t^{\frac{5}{2}}}{\frac{5}{2}} - \frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^5 = \frac{\pi}{16}\left(\frac{2}{5}\left(5^{\frac{5}{2}}-1\right) - \frac{2}{3}\left(5^{\frac{3}{2}}-1\right)\right)$$