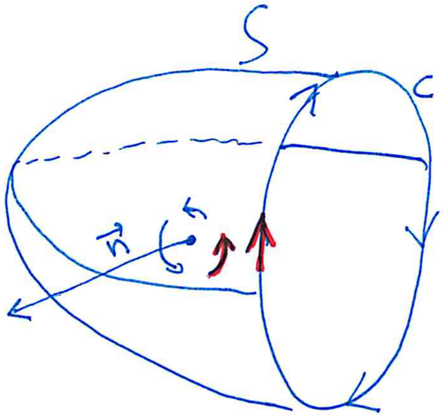
Integrální věty



Orientace okraje plochy c
ve shodě s orientací plochy S

Stokesova věta: Buď S plocha s okrajem orientovaná normálou $\vec{n}$, jejíž okraj tvoří jednoduchá, po částech hladká, uzavřená křivka c orientovaná ve smyslu orientace plochy S.

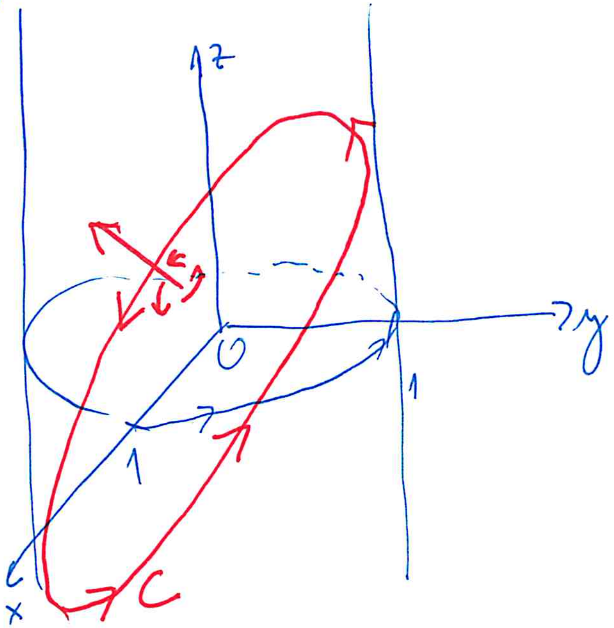
Nechť je dáno vektorové pole $\vec{a}$: $\vec{a}(x,y,z) = u(x,y,z)\cdot\vec{i} + v(x,y,z)\,\vec{j} + w(x,y,z)\,\vec{k}$ a funkce u, v, w mají spojité parciální derivace na S i na c. Pak

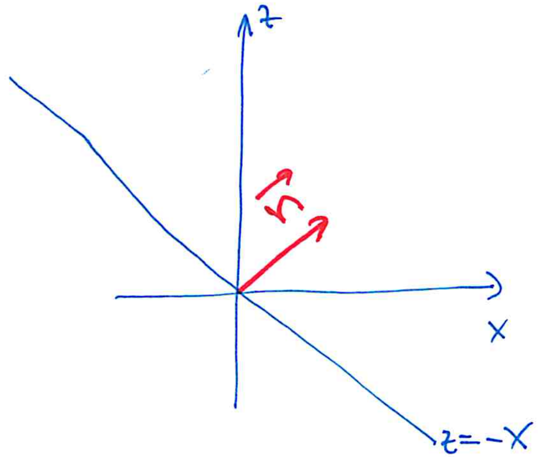
$$\boxed{\int_c \vec{a}\cdot d\vec{r} = \iint_S \operatorname{rot}\vec{a}\cdot\vec{n}\, dS}$$

Příklad: Spočítejte $\int_C \vec{a} \cdot d\vec{r}$, kde $\vec{a} = (\underbrace{y+z}_{u}, \underbrace{x+z}_{v}, \underbrace{x+y}_{w})$ a C

je uzavřená křivka $x^2+y^2=1$ a $x+z=0$ orientovaná souhlasně s $\vec{n}=(1,0,1)$

$z=-x$





$$\int_C \vec{a}\cdot d\vec{r} = \int_C (\underbrace{y+z}_{u})\,dx + (\underbrace{x+z}_{v})\,dy + (\underbrace{x+y}_{w})\,dz =$$

$$= \int_0^{2\pi} \Big[(\sin t - \cos t)\sin t + (\underbrace{\cos t - \cos t}_{0})\cos t +$$

$$+ (\cos t + \sin t)\sin t\Big]\,dt = \int_0^{2\pi} 2\sin t\cos t\,dt =$$

$$= \int_0^{2\pi} \sin 2t\,dt = \underline{\underline{0}}$$

$x = \cos t \quad dx = -\sin t\,dt$

$y = \sin t \quad dy = \cos t\,dt$

$z = -\cos t \quad dz = \sin t\,dt$

$t \in \langle 0, 2\pi \rangle$

Stokesova věta: $\oint_C \vec{a} \cdot d\vec{r} = \iint_S \operatorname{rot} \vec{a} \cdot \vec{n} \, dS = \iint_S \vec{o} \cdot \vec{n} \, dS = \underline{\underline{0}}$

$\vec{a} = (y+z, \ x+z, \ x+y)$

$C: x^2+y^2=1$ a $x+z=0$

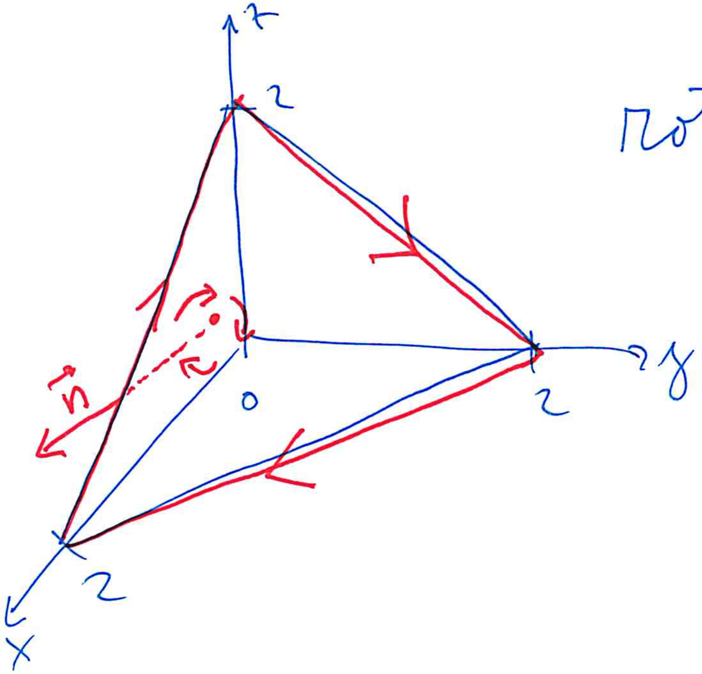
$\vec{n} = (1, 0, 1)$

$$\operatorname{rot} \vec{a} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & x+z & x+y \end{vmatrix} = \vec{\imath}(1-1) + \vec{\jmath}(1-1) + \vec{k}(1-1) = \vec{o}$$

$$\begin{matrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{matrix}$$

(pole $\vec{a}$ je potenciálné $\rightarrow \operatorname{rot} \vec{a} = \vec{o}$)

Příklad: $\displaystyle\int_c \vec{a}\cdot d\vec{r}$ $\quad \vec{a} = (y^2, z^2, x^2)$ a c je obvod Δ s vrcholy $[2,0,0]$, $[0,2,0]$, $[0,0,2]$. Orientace c je souhlasná s $\vec{n} = (-1,-1,-1)$



$$\operatorname{rot}\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x^2 \end{vmatrix} = \vec{i}\cdot 0 + \vec{k}\cdot 0 + \vec{j}\cdot 0 - (\vec{k}\,2y + \vec{i}\,2z + \vec{j}\,2x) =$$

$$\begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{array}$$

$$= (-2z, -2x, -2y)$$

$$\int_C \vec{a}\cdot d\vec{r} = \iint_S \operatorname{rot}\vec{a}\cdot\vec{n}\,dS = \iint_A (-2z,-2x,-2y)\cdot(-1,-1,-1)\,dx\,dy = *$$

plocha: $z = 2 - x - y$

$\vec{a} = (y^2, z^2, x^2)$

$F(x,y,z) = 2 - x - y - z = 0$

$\operatorname{rot}\vec{a} = (-2z, -2x, -2y)$

$\operatorname{Grad} F = (-1,-1,-1)$

$\vec{n} = (-1,-1,-1)$

$[2,0,0], [0,2,0], [0,0,2]$

$$* = \iint_A (4 - \cancel{2x} - \cancel{2y} + \cancel{2x} + \cancel{2y})\,dx\,dy = \iint_A 4\,dx\,dy =$$

$$= 4\iint_A 1\,dx\,dy = 4S_A = 4\cdot 2 = \underline{\underline{8}}$$