Použití LT na řešení OLDR s konst. koef a jejich soustavy

Příklad: $y'' + 4y = \cos t$, $y(0) = 0$, $y'(0) = 0$

$$\mathcal{L}(\cos t) = \frac{s}{s^2+1}$$

$$\mathcal{L}(y) = Y$$

$$\mathcal{L}(y'') = s^2 Y - s\,0 - 0 = s^2 Y$$

obraz rovnice: $s^2 Y + 4Y = \dfrac{s}{s^2+1}$

$$Y(s^2+4) = \frac{s}{s^2+1} \quad / : (s^2+4)$$

$$\underbrace{Y = \frac{s}{(s^2+1)(s^2+4)}}_{\text{obraz řešení}} \longrightarrow y = \frac{\cos t - \cos 2t}{3}$$

$$y'' + 9y = 1 + t + \cos 2t \,, \quad y(0) = 4 \,, \quad y'(0) = -2$$

$$\mathcal{L}(1 + t + \cos 2t) = \frac{1}{s} + \frac{1}{s^2} + \frac{s}{s^2+4}$$

$$\mathcal{L}(y) = Y$$

$$\left(\mathcal{L}(y') = sY - 4\right)$$

$$\mathcal{L}(y'') = s^2 Y - s4 + 2$$

obraz rovnice: $s^2 Y \underset{\sim}{- s4} \underset{\sim}{+2} + 9Y = \frac{1}{s} + \frac{1}{s^2} + \frac{s}{s^2+4}$

$$Y(s^2+9) = \frac{1}{s} + \frac{1}{s^2} + \frac{s}{s^2+4} + 4s - 2 \quad /:(s^2+9)$$

$$Y = \frac{1}{s(s^2+9)} + \frac{1}{s^2(s^2+9)} + \frac{s}{(s^2+4)(s^2+9)} + \frac{4s}{s^2+9} - \frac{2}{s^2+9}$$

$$y = \frac{1 - \cos 3t}{9} + \frac{3t - \sin 3t}{27} + \frac{\cos 2t - \cos 3t}{5} + 4\cos 3t - 2\,\frac{1}{3}\sin 3t$$

$$y'' + 4y' + 4y = t^4 e^{-2t}, \quad y(0) = 0, \quad y'(0) = 0$$

$$\mathcal{L}(y) = Y \qquad\qquad \mathcal{L}(t^4 e^{-2t}) = \frac{4!}{(s+2)^5} = \frac{24}{(s+2)^5}$$

$$\mathcal{L}(y') = sY - 0 = sY$$

$$\mathcal{L}(y'') = s^2 Y - s\cdot 0 - 0 = s^2 Y$$

obraz rovnice:

$$s^2 Y + 4sY + 4Y = \frac{24}{(s+2)^5}$$

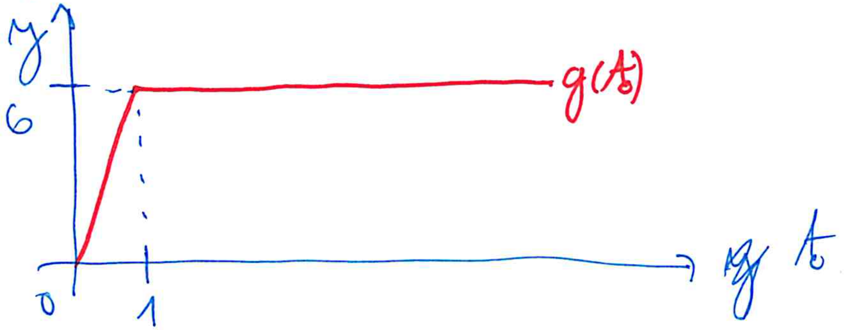
$$Y(s^2 + 4s + 4) = \frac{24}{(s+2)^5}$$

$$Y(s+2)^2 = \frac{24}{(s+2)^5} \quad / : (s+2)^2$$

$$Y = \frac{24}{(s+2)^7} \longrightarrow y = 24\frac{t^6 e^{-2t}}{6!} = \frac{1}{30} t^6 e^{-2t}$$

$$y'' + 36y = g(t), \quad y(0) = 0, \quad y'(0) = -6$$

$g(t) = 6t$ pro $t \in \langle 0, 1\rangle$ a $g(t) = 6$ pro $t \geq 1$



$$g(t) = 6t - 6t\,u(t-1) + 6u(t-1)$$

$$g(t) = 6t - 6u(t-1)(t-1)$$

$$\mathcal{L}(g(t)) = \frac{6}{s^2} - 6\frac{1}{s^2}e^{-s}$$

$$\mathcal{L}(y) = Y, \quad \mathcal{L}(y'') = s^2 Y + 6$$

obraz rovnice:

$$s^2 Y + 6 + 36Y = \frac{6}{s^2} - \frac{6}{s^2}e^{-s}$$

$$Y(s^2 + 36) = \frac{6}{s^2} - \frac{6}{s^2}e^{-s} - 6 \quad / :(s^2+36)$$

$$Y = \frac{6}{s^2(s^2+36)} - \frac{6}{s^2(s^2+36)}e^{-s} - \frac{6}{s^2+36}$$

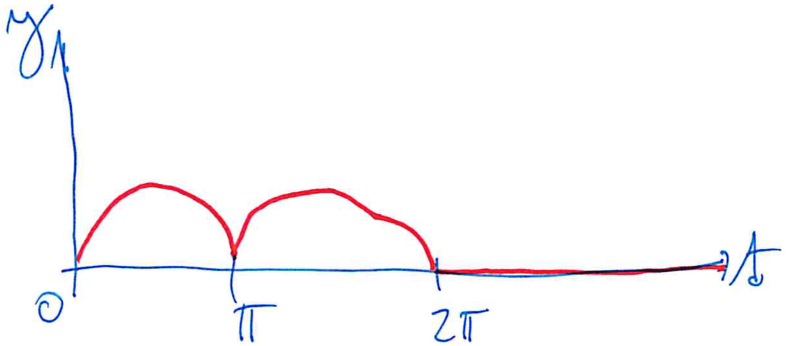
$$y = 6\,\frac{6t - \sin 6t}{6^3} - 6\,\frac{6(t-1) - \sin 6(t-1)}{6^3}\,u(t-1) - 6\,\frac{1}{6}\sin 6t$$

$$y'' + 9y = f(t), \quad y(0) = -2, \quad y'(0) = 3,$$

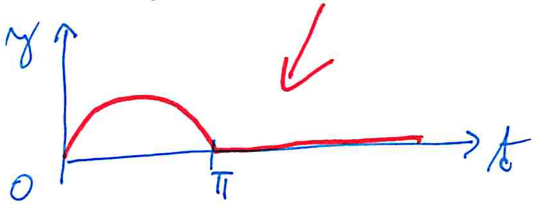
$f(t) = \sin t$ pro $t \in \langle 0, \pi)$

$f(t) = -\sin t$ pro $t \in \langle \pi, 2\pi)$

$f(t) = 0$ pro $t \geq 2\pi$

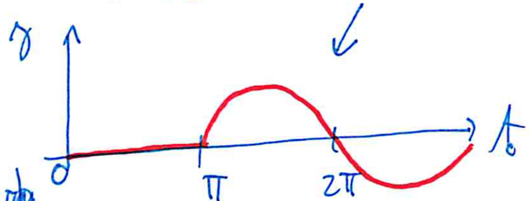


$\sin t - \sin t \, u(t-\pi)$



$$f(t) = \sin t - \sin t \, u(t-\pi) - \sin t \, u(t-\pi) + \sin t \, u(t-2\pi)$$

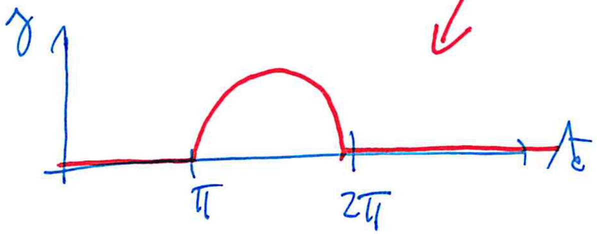
$-\sin t \cdot u(t-\pi)$



$$f(t) = \sin t - 2\sin t \, u(t-\pi) + \sin t \, u(t-2\pi)$$

$$f(t) = \sin t + 2\sin(t-\pi)u(t-\pi) + \sin(t-2\pi)u(t-2\pi)$$

$-\sin t \cdot u(t-\pi) + \sin t \, u(t-2\pi)$



$$\mathcal{L}(f(t)) = \frac{1}{s^2+1} + 2\frac{1}{s^2+1}e^{-\pi s} + \frac{1}{s^2+1}e^{-2\pi s}$$

$$y'' + 9y = f(t), \quad y(0) = -2, \quad y'(0) = 3$$

$$\mathcal{L}(f(t)) = \frac{1}{s^2+1} + 2\frac{1}{s^2+1}e^{-\pi s} + \frac{1}{s^2+1}e^{-2\pi s}$$

$$\mathcal{L}(y) = Y \qquad \mathcal{L}(y'') = s^2 Y + 2s - 3$$

obraz rownie : $s^2 Y + 2s - 3 + 9Y = \frac{1}{s^2+1} + 2\frac{1}{s^2+1}e^{-\pi s} + \frac{1}{s^2+1}e^{-2\pi s}$

$$Y(s^2+9) = \frac{1}{s^2+1} + 2\frac{1}{s^2+1}e^{-\pi s} + \frac{1}{s^2+1}e^{-2\pi s} - 2s + 3$$

$$Y = \frac{1}{(s^2+1)(s^2+9)} + 2\frac{1}{(s^2+1)(s^2+9)}e^{-\pi s} + \frac{1}{(s^2+1)(s^2+9)}e^{-2\pi s} - \frac{2s}{s^2+9} + \frac{3}{s^2+9}$$

$$y = \frac{3\sin t - \sin 3t}{3\cdot 8} + 2\,\frac{3\sin(t-\pi) - \sin 3(t-\pi)}{24}u(t-\pi) + \frac{3\sin(t-2\pi) - \sin 3(t-2\pi)}{24}u(t-2\pi)$$

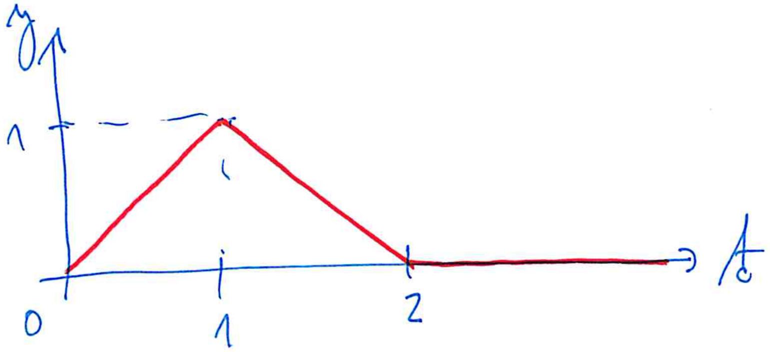
$$-2\cos 3t + \sin 3t$$

$y'' + 9y = h(t)$, $y(0)=0$, $y'(0)=0$,

$h(t) = t$, $t \in \langle 0,1)$

$h(t) = 2-t$, $t \in \langle 1,2)$

$h(t) = 0$, $t \geq 2$

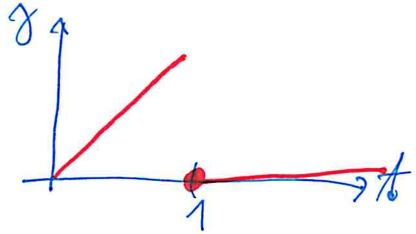


$$h(t) = t - t\,\underline{u(t-1)} + (2-t)\,\underline{u(t-1)} - (\underline{2-t})\,u(\underline{t-2})$$

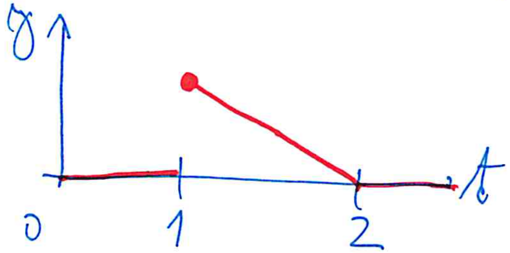
$$h(t) = t - 2(t-1)\,u(t-1) + (t-2)\,u(t-2)$$

$$\mathcal{L}(h(t)) = \frac{1}{s^2} - \frac{2}{s^2}e^{-s} + \frac{1}{s^2}e^{-2s}$$

$h_1(t) = t - t\,u(t-1)$



$h_2(t) = (2-t)\,u(t-1) - (2-t)\,u(t-2)$



$$y'' + 9y = h(t),\ y(0) = 0,\ y'(0) = 0$$

$$\mathcal{L}(h(t)) = \frac{1}{s^2} - \frac{2}{s^2}e^{-s} + \frac{1}{s^2}e^{-2s}$$

$$\mathcal{L}(y) = Y,\quad \mathcal{L}(y'') = s^2 Y$$

obraz rovnice:

$$s^2 Y + 9Y = \frac{1}{s^2} - \frac{2}{s^2}e^{-s} + \frac{1}{s^2}e^{-2s}$$

$$Y = \frac{1}{s^2(s^2+9)} - \frac{2}{s^2(s^2+9)}e^{-s} + \frac{1}{s^2(s^2+9)}e^{-2s}$$

$$\downarrow$$

$$y = \frac{3t - \sin 3t}{27} - 2\frac{3(t-1) - \sin 3(t-1)}{27}u(t-1) + \frac{3(t-2) - \sin 3(t-2)}{27}u(t-2)$$

$$-y_1' + y_2 = e^{-3t} \qquad y_1(0)=1,\ y_2(0)=-2$$
$$y_2' - 3y_1 = 3e^{-3t}$$

$$\mathcal{L}(y_1) = Y_1,\quad \mathcal{L}(y_2) = Y_2,\quad \mathcal{L}(y_1') = sY_1 - 1,\quad \mathcal{L}(y_2') = sY_2 + 2$$

$$\mathcal{L}(e^{-3t}) = \frac{1}{s+3}$$

obraz soustavy:

$$1 - sY_1 + Y_2 = \frac{1}{s+3} \to Y_2 = \frac{1}{s+3} + sY_1 - 1 = \frac{1}{s+3} + \frac{s}{s+3} - 1 = \frac{1+s-s-3}{s+3} =$$
$$= -2\cdot\frac{1}{s+3} \to \boxed{y_2 = -2e^{-3t}}$$

$$2 + sY_2 - 3Y_1 = \frac{3}{s+3}$$

$$2 + \frac{s}{s+3} + s^2 Y_1 - s - 3Y_1 = \frac{3}{s+3}$$

$$Y_1(\cancel{s^2-3}) = \frac{3}{s+3} - \frac{s}{s+3} + s - 2 = \frac{3-s}{s+3} + s - 2 = \frac{3-\cancel{s}+s^2+3\cancel{s}-2\cancel{s}-6}{s+3} = \frac{\cancel{s^2-3}}{s+3}$$

$$Y_1 = \frac{1}{s+3} \to \boxed{y_1 = e^{-3t}}$$

$$y_1'' - y_2'' + y_2' - y_1 = e^t - 2 \qquad y_1(0)=0,\ y_1'(0)=0 \qquad \boxed{\text{chceme } y_1}$$

$$2y_1'' - y_2'' - 2y_1' + y_2 = -t \qquad y_2(0)=0,\ y_2'(0)=0$$

$$\mathcal{L}(y_1) = Y_1,\ \mathcal{L}(y_1') = sY_1,\ \mathcal{L}(y_1'') = s^2 Y_1,\ \mathcal{L}(e^t - 2) = \frac{1}{s-1} - \frac{2}{s}$$

$$\mathcal{L}(y_2) = Y_2,\ \mathcal{L}(y_2') = sY_2,\ \mathcal{L}(y_2'') = s^2 Y_2,\ \mathcal{L}(-t) = -\frac{1}{s^2}$$

obraz soustavy:

$$s^2 Y_1 - s^2 Y_2 + sY_2 - Y_1 = \frac{1}{s-1} - \frac{2}{s}$$

$$2s^2 Y_1 - s^2 Y_2 - 2sY_1 + Y_2 = -\frac{1}{s^2}$$

$$Y_1 \underbrace{(s^2-1)}_{(s-1)(s+1)} + Y_2 \underbrace{(s-s^2)}_{s(1-s)} = \frac{1}{s-1} - \frac{2}{s} = \frac{2-s}{s(s-1)} \quad /:(s-1)$$

$$Y_1 \underbrace{(2s^2-2s)}_{2s(s-1)} + Y_2 \underbrace{(1-s^2)}_{(1-s)(1+s)} = -\frac{1}{s^2} \quad /:(s-1)$$

$$Y_1(s+1) - Y_2 s = \frac{2-s}{s(s-1)^2} \qquad /\cdot(1+s)$$

$$Y_1 2s - Y_2(1+s) = -\frac{1}{s^2(s-1)} \qquad /\cdot(-s)$$

$$Y_1\left[(1+s)^2 - 2s^2\right] = \frac{(2-s)(1+s)}{s(s-1)^2} + \frac{\cancel{s}}{s^{\cancel{2}}(s-1)} = \frac{2+s-s^2+s-1}{s(s-1)^2}$$

$$Y_1(\cancel{-s^2+2s+1}) = \frac{\cancel{-s^2+2s+1}}{s(s-1)^2}$$

$$(s-1)^2 = s^2-2s+1$$

$$Y_1 = \frac{1}{s(s-1)^2} = \frac{A}{s} + \frac{B}{(s-1)^2} + \frac{C}{s-1} = \frac{A(s^2-2s+1) + Bs + C(s^2-s)}{s(s-1)^2}$$

u s^2: $0 = A + C$ $\qquad \underline{C = -1}$

u s: $0 = -2A + B - C$ $\qquad \underline{B = 1}$

u s^0: $1 = A \rightarrow \underline{A = 1}$

$$Y_1 = \frac{1}{s} + \frac{1}{(s-1)^2} - \frac{1}{s-1}$$

$$\boxed{y_1 = 1 + te^t - e^t}$$