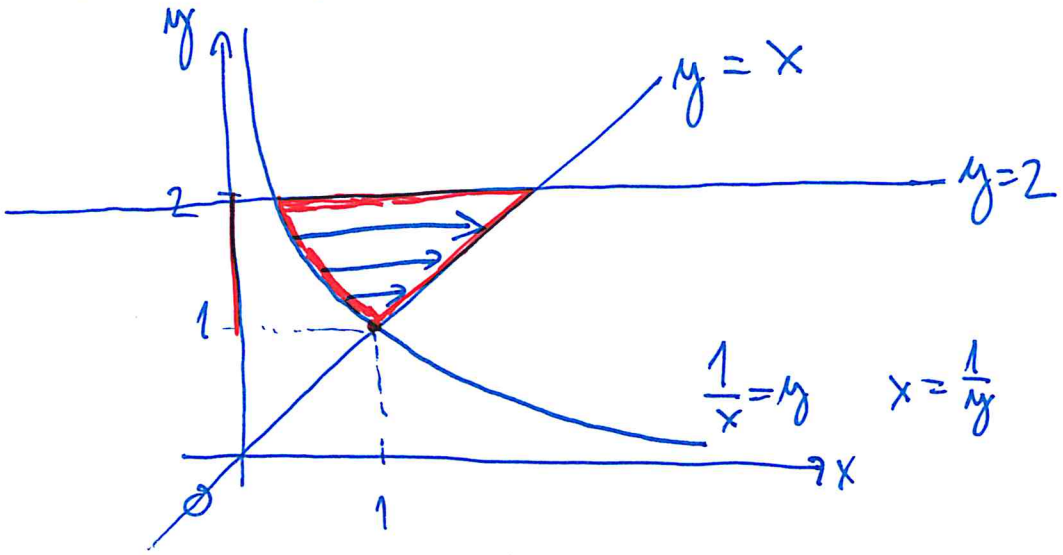
1) Spočítejte obsah A: $y = \frac{1}{x}$, $y = x$, $y = 2$

$$S = \iint_A 1 \, dx\,dy$$



$$1 \le y \le 2$$
$$\frac{1}{y} \le x \le y$$

$$S = \iint_A 1 \, dx\,dy = \int_1^2 \left(\int_{\frac{1}{y}}^{y} 1 \, dx \right) dy$$

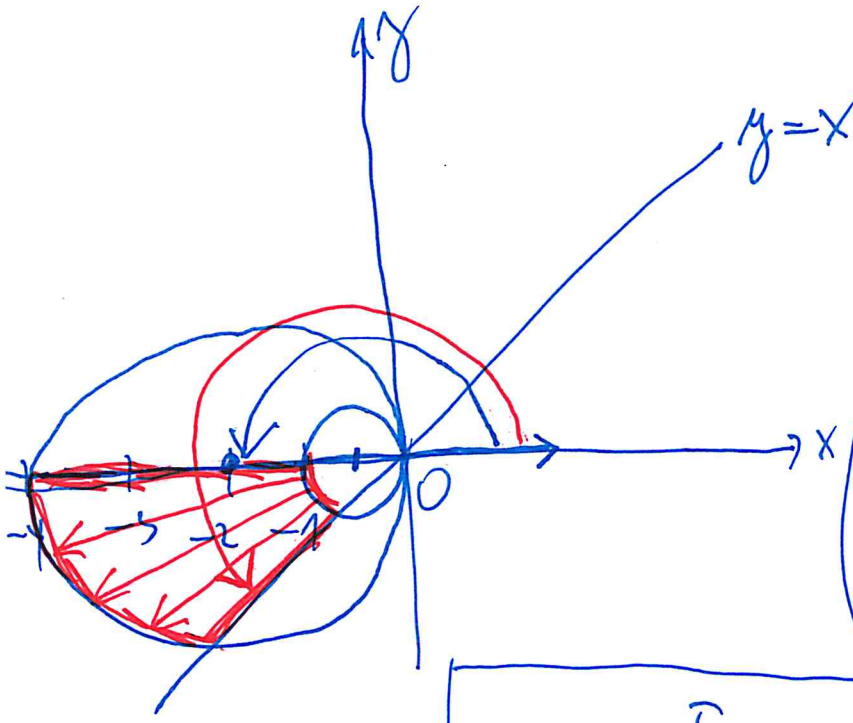
$$\int_{\frac{1}{y}}^{y} 1 \, dx = \left[x \right]_{\frac{1}{y}}^{y} = y - \frac{1}{y}$$

$$= \int_1^2 \left(y - \frac{1}{y} \right) dy = \left[\frac{y^2}{2} - \ln y \right]_1^2 =$$

$$= 2 - \ln 2 - \frac{1}{2} = \frac{3}{2} - \ln 2$$

Obsah plochy Bohraničené $x^2+y^2+x=0$, $y=x$, $y=0$

$$x^2+y^2+4x=0$$



MENŠÍ KRUŽNICE

$$x^2+y^2+x=0$$

$$x^2+x+y^2=0$$

$$\left(x+\frac{1}{2}\right)^2+y^2=\frac{1}{4}$$

VĚTŠÍ KRUŽNICE

$$x^2+y^2+4x=0$$

$$(x+2)^2+y^2=4$$

$$\rho^2\cos^2\varphi+\rho^2\sin^2\varphi+4\rho\cos\varphi=0$$

$$x=\rho\cos\varphi$$

$$y=\rho\sin\varphi$$

$$\pi\le\varphi\le\frac{5}{4}\pi$$

$$-\cos\varphi\le\rho\le-4\cos\varphi$$

$$\rho^2\cos^2\varphi+\rho^2\sin^2\varphi+\rho\cos\varphi=0$$

$$\rho^2+\rho\cos\varphi=0$$

$$\rho(\rho+\cos\varphi)=0$$

$$\rho=-\cos\varphi$$

$$\rho^2+4\rho\cos\varphi=0$$

$$\rho(\rho+4\cos\varphi)=0$$

$$\rho=-4\cos\varphi$$

$$S(B) = \iint_B 1 \, dx\,dy = \iint_B \varrho \, d\varrho\, d\varphi = \int_{\pi}^{\frac{5}{4}\pi} \left(\int_{-\cos\varphi}^{-4\cos\varphi} \varrho \, d\varrho \right) d\varphi = \int_{\pi}^{\frac{5}{4}\pi} \frac{15}{4}(1+\cos 2\varphi)\, d\varphi =$$

$t = 2\varphi$

$dt = 2\,d\varphi$

$\frac{1}{2}dt = d\varphi$

$$\pi \le \varphi \le \frac{5}{4}\pi$$

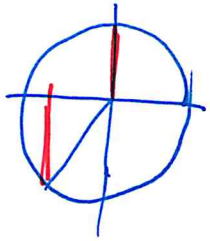
$$-\cos\varphi \le \varrho \le -4\cos\varphi$$

$$= \frac{15}{4}\left[\varphi + \sin(2\varphi)\cdot\frac{1}{2}\right]_{\pi}^{\frac{5}{4}\pi} = \frac{15}{4}\left(\frac{5}{4}\pi + \frac{1}{2}(\quad) - (\pi + 0)\right) =$$

$$= \frac{15}{4}\left(\frac{\pi}{4} + \frac{1}{2}\right) \doteq 4{,}8$$

$x = \varrho\cos\varphi$

$y = \varrho\sin\varphi$



$dx\,dy \to \varrho\,d\varrho\,d\varphi$

$$\int_{-\cos\varphi}^{-4\cos\varphi} \varrho \, d\varrho = \left[\frac{1}{2}\varrho^2\right]_{-\cos\varphi}^{-4\cos\varphi} = \frac{1}{2}\left(16\cos^2\varphi - \cos^2\varphi\right) = \frac{15}{2}\cos^2\varphi = \underline{\frac{15}{4}(1+\cos 2\varphi)}$$

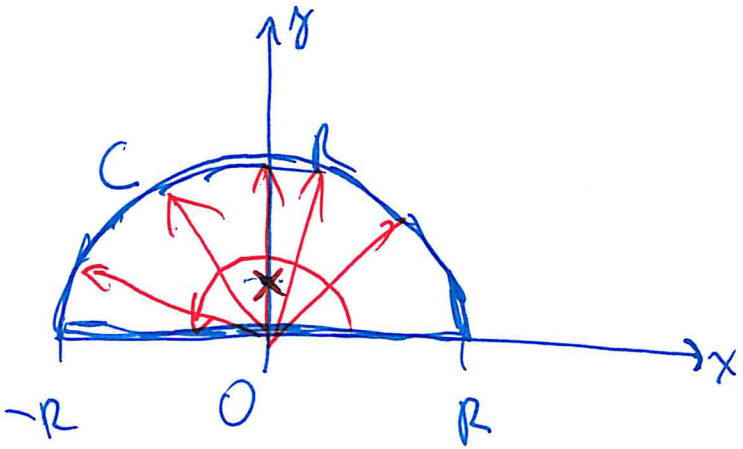
$$\cos 2\varphi = \cos^2\varphi - \sin^2\varphi = \cos^2\varphi - (1-\cos^2\varphi) = 2\cos^2\varphi - 1$$

$$\cos^2\varphi = \frac{1}{2}(1+\cos 2\varphi)$$

Nalezněte souřadnice těžiště $h(x,y)=1$

$x^2+y^2 \le R^2, \; y \ge 0$

$$\boxed{x_T = 0}$$



$$y_T = \frac{1}{m(C)} \iint_C y \cdot 1 \, dx\,dy = \frac{2}{\pi R^2} \iint_C y \, dx\,dy =$$

$$m = h \cdot S = 1 \cdot \frac{1}{2}\pi R^2$$

$$= \frac{2}{\pi R^2} \cdot \frac{2}{3} R^3 = \boxed{\frac{4R}{3\pi} = y_T}$$

$0 \le \varphi \le \pi$

$0 \le \rho \le R$

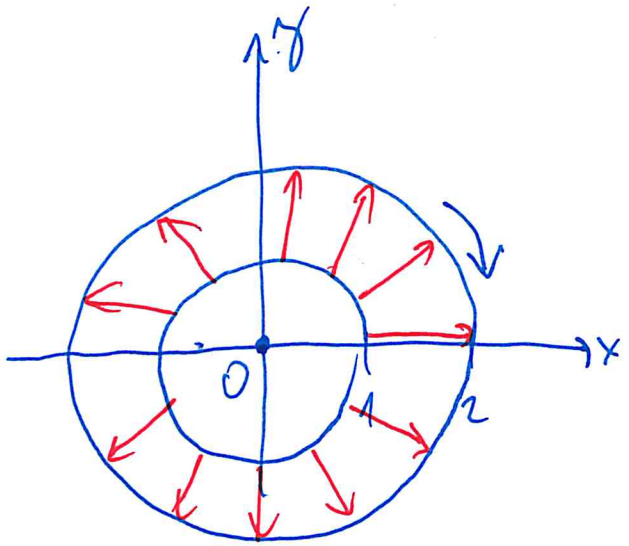
$x = \rho\cos\varphi$

$y = \rho\sin\varphi$

$$\iint_C y \, dx\,dy = \iint \rho^2 \sin\varphi \, d\rho\,d\varphi = \int_0^{\pi} \left(\int_0^R \rho^2 \sin\varphi \, d\rho \right) d\varphi =$$

$$= \int_0^{\pi} \left[\frac{\rho^3}{3} \right]_0^R \sin\varphi \, d\varphi = \frac{R^3}{3} \int_0^{\pi} \sin\varphi \, d\varphi = \frac{R^3}{3} \left[-\cos\varphi \right]_0^{\pi} = \frac{R^3}{3}(1+1) = \frac{2}{3}R^3$$

Moment setrvačnosti $D: x^2+y^2=1$ $\quad x^2+y^2=4$ $\quad h(x,y)=h$



$$I = \iint_D (x^2+y^2)\cdot h(x,y)\,dx\,dy = h\iint_D (x^2+y^2)\,dx\,dy =$$

$$= h\cdot \iint_D (\varrho^2\cos^2\varphi + \varrho^2\sin^2\varphi)\cdot \varrho\,d\varrho\,d\varphi = h\iint_D \varrho^3\,d\varrho\,d\varphi =$$

$$= h\int_0^{2\pi}\left(\int_1^2 \varrho^3\,d\varrho\right)d\varphi = h\int_0^{2\pi}\left[\frac{\varrho^4}{4}\right]_1^2 d\varphi =$$

$$= h\cdot\int_0^{2\pi}\left(4-\frac{1}{4}\right)d\varphi = \frac{15}{4}h\,[\varphi]_0^{2\pi} = \boxed{\frac{15h\pi}{2}}$$

$0 \le \varphi \le 2\pi$ $\quad x=\varrho\cos\varphi$

$1 \le \varrho \le 2$ $\quad y=\varrho\sin\varphi$

$x^2+y^2=2x$ $\quad x^2+y^2=4x$

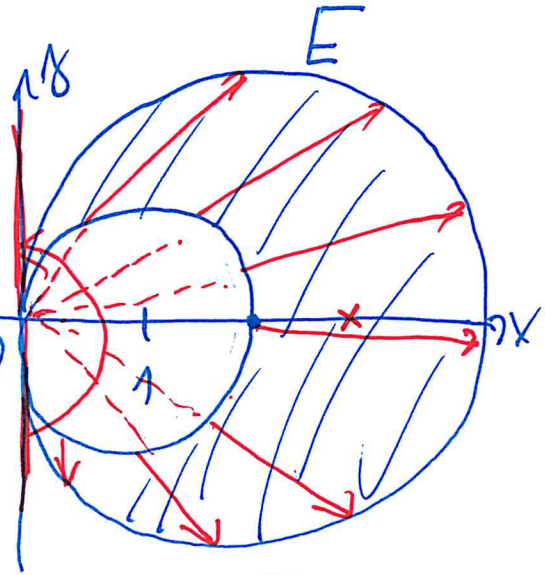
$x^2-2x+y^2=0$ $\quad x^2-4x+y^2=0$

$(x-1)^2+y^2=1$ $\quad (x-2)^2+y^2=4$

Těžiště $\quad h(x,y)=1$

$$\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$$

$$2\cos\varphi \le \rho \le 4\cos\varphi$$



$x=\rho\cos\varphi$

$y=\rho\sin\varphi$

$\rho^2\cos^2\varphi+\rho^2\sin^2\varphi-2\rho\cos\varphi=0$

$\rho^2-2\rho\cos\varphi=0$

$\rho(\rho-2\cos\varphi)=0$

$\boxed{\rho=2\cos\varphi}$

$\rho^2-4\rho\cos\varphi=0$

$\rho(\rho-4\cos\varphi)=0$

$\boxed{\rho=4\cos\varphi}$

$\boxed{y_T=0}$

$$x_T=\frac{1}{m}\iint_E x\,dx\,dy=\frac{1}{3\pi}\iint_E x\,dx\,dy$$

$m=h\cdot S=\pi\not4-\pi=3\pi$

$$x_T = \frac{1}{3\pi}\iint_E x\,dx\,dy = \frac{1}{3\pi}\iint_E \rho^2\cos\varphi\,d\rho\,d\varphi = \frac{1}{3\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\int_{2\cos\varphi}^{4\cos\varphi}\rho^2\cos\varphi\,d\rho\right)d\varphi =$$

$$-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$$

$$2\cos\varphi \le \rho \le 4\cos\varphi$$

$$= \frac{1}{3\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{56}{3}\cos^4\varphi\,d\varphi = \frac{56}{9\pi}\cdot\frac{3\pi}{8} = \frac{7}{3}$$

$$x = \rho\cos\varphi$$

$$y = \rho\sin\varphi$$

$$\int_{2\cos\varphi}^{4\cos\varphi}\rho^2\cos\varphi\,d\rho = \cos\varphi\,\frac{1}{3}\left[\rho^3\right]_{2\cos\varphi}^{4\cos\varphi} = \frac{1}{3}\cos\varphi\left(64\cos^3\varphi - 8\cos^3\varphi\right) = \frac{56}{3}\cos^4\varphi$$