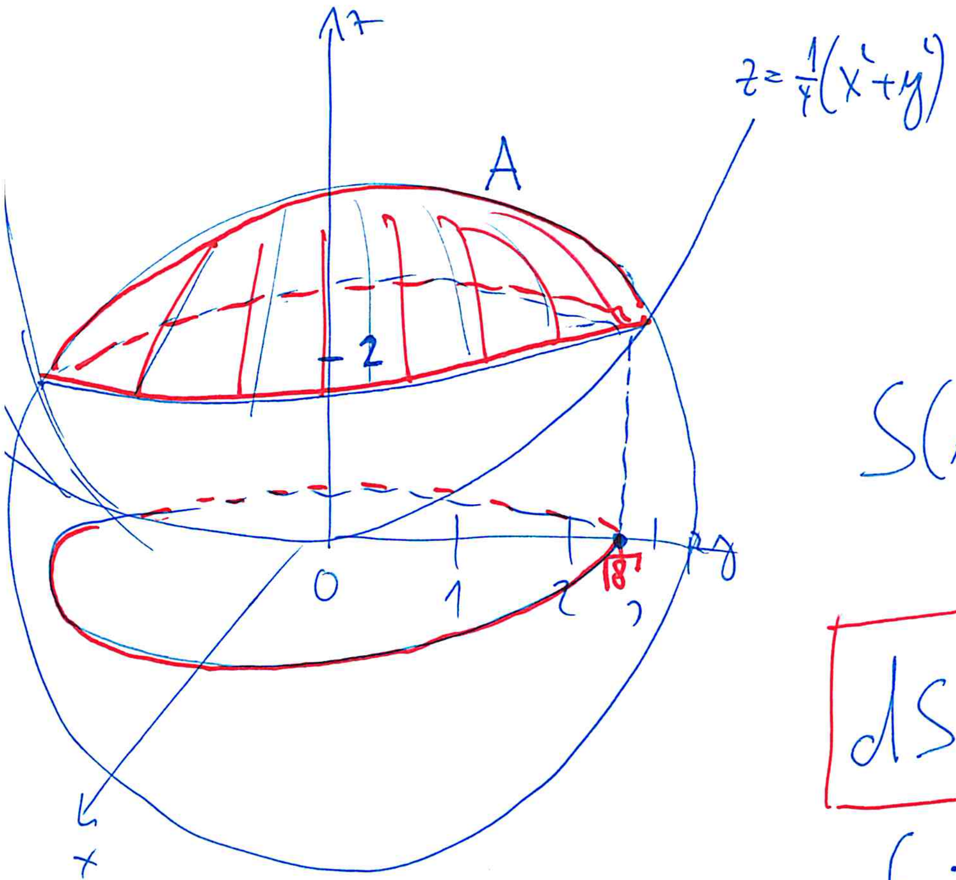
1) Spočítejte obsah sféry $x^2+y^2+z^2=12$ uvnitř $4z=x^2+y^2$



$4z+z^2=12 \rightarrow z^2+4z-12=0$

$(z+6)(z-2)=0$

$\emptyset \quad \boxed{z=2}$

$x^2+y^2=8$

$$S(A)=\iint_A 1\,dS$$

$$\boxed{dS=\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dx\,dy}$$

$f: \ z=\sqrt{12-x^2-y^2}$

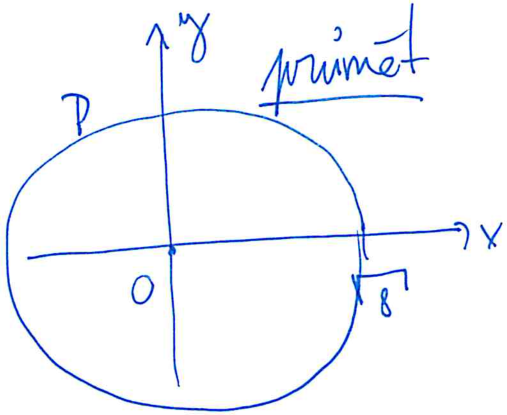
$$\frac{\partial f}{\partial x}=\frac{-\cancel{2}x}{\cancel{2}\sqrt{12-x^2-y^2}}$$

$$\frac{\partial f}{\partial y}=\frac{-\cancel{2}y}{\cancel{2}\sqrt{12-x^2-y^2}}$$

$$dS=\sqrt{1+\frac{x^2}{12-x^2-y^2}+\frac{y^2}{12-x^2-y^2}}\,dx\,dy=\sqrt{\frac{12}{12-x^2-y^2}}\,dx\,dy$$

$$S(A)=\iint_A 1\,dS=\iint_P\sqrt{\frac{12}{12-x^2-y^2}}\,dx\,dy=\sqrt{12}\iint_P\frac{\rho}{\sqrt{12-\rho^2}}\,d\rho\,d\varphi=$$

$$dS=\sqrt{\frac{12}{12-x^2-y^2}}\,dx\,dy$$



$$x=\rho\cos\varphi$$
$$y=\rho\sin\varphi$$
$$0\le\varphi\le 2\pi$$
$$0\le\rho\le\sqrt{8}$$

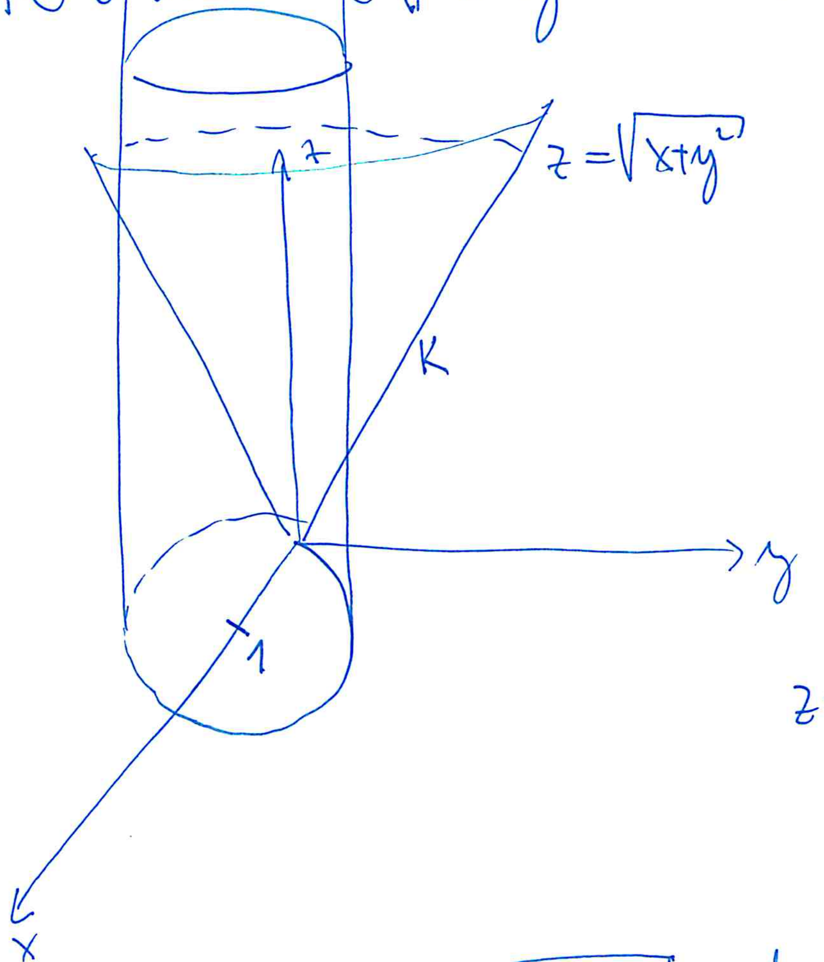
$$=\sqrt{12}\int_0^{2\pi}\Bigg(\underbrace{\int_0^{\sqrt{8}}\frac{\rho}{\sqrt{12-\rho^2}}\,d\rho}\Bigg)\,d\varphi=$$

$$=\underbrace{\sqrt{12}}_{2\sqrt{3}}\,(\underbrace{2\sqrt{3}}_{\sqrt{12}}-2)\int_0^{2\pi}1\,d\varphi=(12-4\sqrt{3})\cdot 2\pi$$

$$\int_0^{\sqrt{8}}\frac{\rho}{\sqrt{12-\rho^2}}\,d\rho=-\frac{1}{2}\int_{12}^{4}\frac{1}{\sqrt{t}}\,dt=\frac{\cancel{1}}{\cancel{2}}\left[\frac{t^{\frac{1}{2}}}{\cancel{\frac{1}{2}}}\right]_4^{12}=\sqrt{12}-2=2\sqrt{3}-2$$

$$t=12-\rho^2$$
$$dt=-2\rho\,d\rho\rightarrow-\frac{1}{2}dt=\rho\,d\rho$$

Těžiště $z=\sqrt{x^2+y^2}$ ležící uvnitř $x^2+y^2-2x=0$. $h(x,y,z)=1$



$$(x-1)^2+y^2=1$$

$\boxed{y_T=0}$ $x_T=?$ $z_T=?$ $m=h\cdot \overset{?}{S}$

$$m=h\cdot S=\iint_K 1\,dS$$

$$z=f(x,y)=\sqrt{x^2+y^2}$$

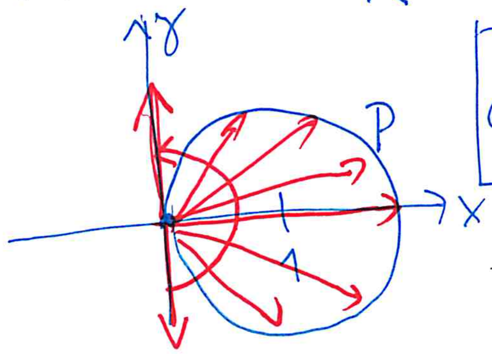
$$\frac{\partial f}{\partial x}=\frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial y}=\frac{y}{\sqrt{x^2+y^2}}$$

$$dS=\sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}\,dxdy=\underline{\sqrt{2}\,dxdy}$$

$$S = \iint_K 1\, dS = \iint_P \sqrt{2}\, dx\,dy = \sqrt{2} \iint_P 1\, dx\,dy = \underline{\sqrt{2}\,\pi = m}$$

Průmět



$$\boxed{dS = \sqrt{2}\, dx\,dy}$$

$x = \rho \cos\varphi$

$y = \rho \sin\varphi$

$$x_T = \frac{1}{m} \iint_K x \, dS = \frac{1}{\sqrt{2}\pi} \cdot \sqrt{2}\pi = \boxed{1 = x_T}$$

$x^2 + y^2 - 2x = 0$

$\rho^2 - 2\rho\cos\varphi = 0$

$\rho(\rho - 2\cos\varphi) = 0$

$\rho = 0 \qquad \boxed{\rho = 2\cos\varphi}$

$$\boxed{\begin{array}{l} -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2} \\ 0 \le \rho \le 2\cos\varphi \end{array}}$$

$$\iint_K x\, dS = \iint_P x\sqrt{2}\, dx\,dy = \sqrt{2} \iint \rho^2 \cos\varphi \, d\rho\, d\varphi =$$

$$= \sqrt{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_0^{2\cos\varphi} \rho^2 \cos\varphi \, d\rho \right) d\varphi = \sqrt{2} \cdot \frac{8}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\varphi \, d\varphi = \underline{\sqrt{2}\, \frac{\cancel{8}}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{8}} \pi}$$

$$\int_0^{2\cos\varphi} \rho^2 \cos\varphi \, d\rho = \cos\varphi \left[\frac{\rho^3}{3} \right]_0^{2\cos\varphi} = \frac{8}{3} \cos^4\varphi$$

$$z_T = \frac{1}{m}\iint_K z\,dS = \frac{1}{\cancel{\sqrt{2}}\,\pi}\iint_P \overset{(z)}{\sqrt{x^2+y^2}}\,\cancel{\sqrt{2}}\,dx\,dy = \frac{1}{\pi}\iint_P \varrho^2\,d\varrho\,d\varphi =$$

$m = \sqrt{2}\,\pi$

$dS = \sqrt{2}\,dx\,dy$

$$\boxed{z = \sqrt{x^2+y^2}}$$

$-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$

$0 \le \varrho \le 2\cos\varphi$

$$= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\int_0^{2\cos\varphi}\varrho^2\,d\varrho\right)d\varphi = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\frac{\varrho^3}{3}\right]_0^{2\cos\varphi} =$$

$$= \frac{8}{3\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\underbrace{\cos^3\varphi}_{(1-\sin^2\varphi)\cdot\cos\varphi}\,d\varphi = \frac{8}{3\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1-\sin^2\varphi)\cdot\cos\varphi\,d\varphi =$$

$t = \sin\varphi$

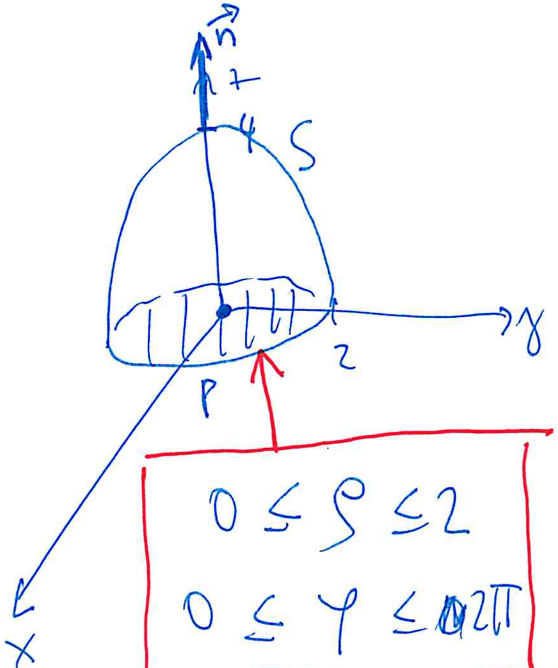
$dt = \cos\varphi\,d\varphi$

$$= \frac{8}{3\pi}\int_{-1}^{1}(1-t^2)\,dt = \frac{8}{3\pi}\left[t-\frac{t^3}{3}\right]_{-1}^{1} = \frac{8}{3\pi}\left(1-\frac{1}{3}-\left(-1+\frac{1}{3}\right)\right) = \frac{8}{3\pi}\cdot\frac{4}{3} = \frac{32}{9\pi}$$

$$\frac{32}{9\pi} = z_T$$

Tok pole $\vec{a}=(y,-x,z)$ paraboloidem $z=4-x^2-y^2$, $z\geq 0$

Plocha je orientovaná $\vec{n}=(0,0,1)$ nad $[0,0]$



$$0 \leq \rho \leq 2$$
$$0 \leq \varphi \leq 2\pi$$

$$\boxed{\iint_S \vec{a}\cdot\vec{n}\,dS = \iint_P \vec{a}\cdot \operatorname{grad} F\,dx\,dy}$$

$F_1(x,y,z)=4-x^2-y^2-z=0 \qquad \operatorname{grad} F_1=(-2x,-2y,-1)$

$F_2(x,y,z)=x^2+y^2+z-4=0 \qquad \operatorname{grad} F_2=(2x,2y,1)$

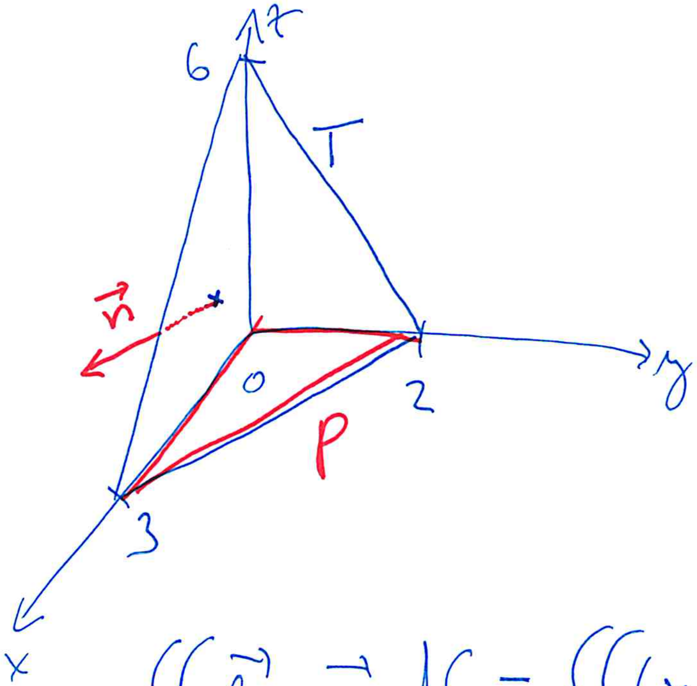
$$\iint_S \vec{a}\cdot\vec{n}\,dS = \iint_P (y,-x,\underbrace{4-x^2-y^2}_{z})\cdot(2x,2y,1)\,dx\,dy = \iint_P (\cancel{2xy}-\cancel{2xy}+$$

$$+4-x^2-y^2)\,dx\,dy = \iint_P (4-x^2-y^2)\,dx\,dy = \iint_P (4-\rho^2)\rho\,d\rho\,d\varphi =$$

$$= \int_0^{2\pi} \left(\int_0^2 (4\varrho - \varrho^3)\, d\varrho \right) d\varphi = \int_0^{2\pi} 4\, d\varphi = \underline{\underline{8\pi}}$$

$$\int_0^2 (4\varrho - \varrho^3)\, d\varrho = \left[2\varrho^2 - \frac{\varrho^4}{4} \right]_0^2 = 8 - 4 = 4$$

Tok $\vec{b} = (x, y-z, 2z)$ plochou Δ $[3,0,0], [0,2,0], [0,0,6]$, $\vec{n} = -(2,3,1)$



$\boxed{2x + 3y + z = 6}$ $\boxed{z = 6 - 2x - 3y}$

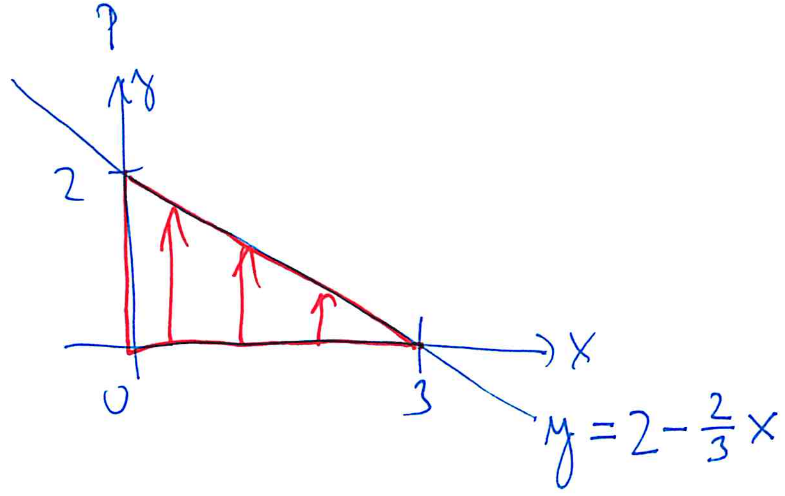
$F(x,y,z) = 6 - 2x - 3y - z = 0$

$\operatorname{grad} F = (-2, -3, -1)$

$$\iint_T \vec{b}\cdot\vec{n}\, dS = \iint_P (x, y-z, 2z)\cdot(-2,-3,-1)\, dx\,dy = \iint_P \big(-2x - 3y + \underbrace{3z - 2z}_{z}\big)\, dx\,dy$$

$$= \iint_P (-2x - 3y + 6 - 2x - 3y)\, dx\,dy = \iint_P (6 - 4x - 6y)\, dx\,dy$$

$$\iint (6-4x-6y)\,dx\,dy = \int_0^3 \left(\int_0^{2-\frac{2}{3}x} (6-4x-6y)\,dy \right) dx =$$



$$= \int_0^3 \left(-4x + \frac{4}{3}x^2\right) dx = \left[-2x^2 + \frac{4}{9}x^3\right]_0^3 = -18+12 =$$

$$= -6$$

$0 \le x \le 3$

$0 \le y \le 2 - \frac{2}{3}x$

$$\int_0^{2-\frac{2}{3}x} (6-4x-6y)\,dy = \left[6y - 4xy - 3y^2\right]_0^{2-\frac{2}{3}x} = 12 - 4x - 8x + \frac{8}{3}x^2 -$$

$$-3\left(4 - \frac{8}{3}x + \frac{4}{9}x^2\right) = -4x + \frac{4}{3}x^2$$