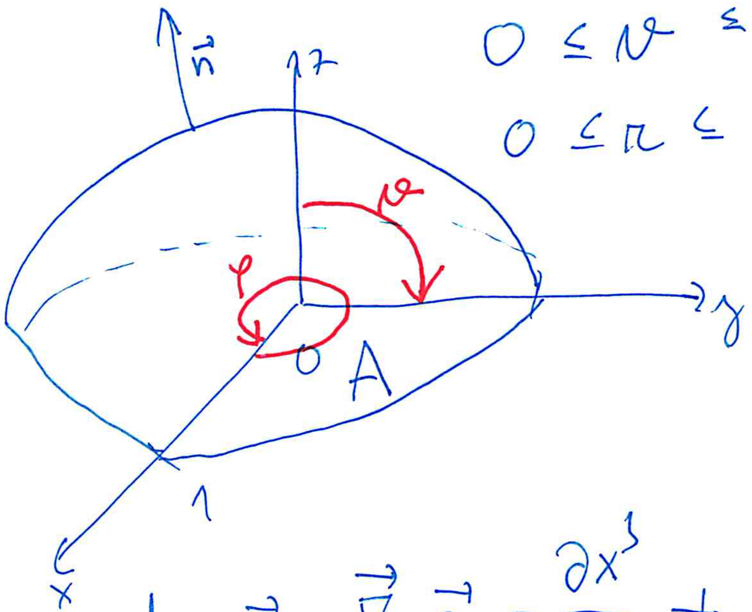
Spočítejte tok vektorového pole $\vec{a} = (x^3, y^3, z^3)$ vnější stranou uzavřené plochy S: horní polosféra $x^2+y^2+z^2=1$ uzavřená zdola $x^2+y^2 \leq 1$ pro $z=0$

$0 \leq \varphi \leq 2\pi$

$0 \leq \vartheta \leq \frac{\pi}{2}$

$0 \leq r \leq 1$



$$\Phi = \iint_S \vec{a} \cdot \vec{n} \, dS = \iiint_A \operatorname{div} \vec{a} \, dx\,dy\,dz =$$

$$= 3 \iiint_A (x^2+y^2+z^2) \, dx\,dy\,dz = 3 \iiint_A r^2 r^2 \sin\vartheta \, dr \, d\varphi \, d\vartheta$$

$$\operatorname{div} \vec{a} = \vec{\nabla} \cdot \vec{a} = \frac{\partial x^3}{\partial x} + \frac{\partial y^3}{\partial y} + \frac{\partial z^3}{\partial z} = 3x^2+3y^2+3z^2$$

$x = r \cos\varphi \sin\vartheta$

$y = r \sin\varphi \sin\vartheta$

$z = r \cos\vartheta$

$$\Phi = 3\iiint_A r^4 \sin\vartheta \, dr \, d\varphi \, d\vartheta = 3 \cancel{\iiint_A} \int_0^{2\pi} \left[\int_0^{\frac{\pi}{2}} \left(\int_0^1 r^4 \sin\vartheta \, dr \right) d\vartheta \right] d\varphi \overset{3)}{=}$$

$$0 \le \varphi \le 2\pi$$
$$0 \le \vartheta \le \frac{\pi}{2}$$
$$0 \le r \le 1$$

$$= 3 \int_0^{2\pi} \frac{1}{5} \, d\varphi = \frac{3}{5} \left[\varphi \right]_0^{2\pi} = \underline{\underline{\frac{6\pi}{5}}}$$

1) $$\int_0^1 r^4 \sin\vartheta \, dr = \left[\frac{r^5}{5} \right]_0^1 \sin\vartheta = \frac{1}{5} \sin\vartheta$$

2) $$\int_0^{\frac{\pi}{2}} \frac{1}{5} \sin\vartheta \, d\vartheta = -\frac{1}{5} \left[\cos\vartheta \right]_0^{\frac{\pi}{2}} = -\frac{1}{5}(0 - 1) = \frac{1}{5}$$

Soustavy OLDR 1. řádu s konst. koeficienty – homogenní

$$\vec{y}\,' = \begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix} \cdot \vec{y}$$

Hledáme funkce $y_1 = y_1(x)$ a $y_2 = y_2(x)$

$$\vec{y} = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}$$

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} \longrightarrow \begin{aligned} y_1'(x) &= -y_1(x) + 3y_2(x) \\ y_2'(x) &= 2y_1(x) - 2y_2(x) \end{aligned}$$

řešení: spočítáme vlastní čísla λ_i a příslušné vlastní vektory $\vec{v}_i$

a řešení soustavy pak je $\vec{y} = e^{\lambda_i x} \cdot \vec{v}_i$

hledáme vlastní čísla a vl. vektory matice $A = \begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix}$

$$0 = \det(A - \lambda E) = \det\left(\begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\begin{bmatrix} -1-\lambda & 3 \\ 2 & -2-\lambda \end{bmatrix}$$

$$0 = (-1-\lambda)(-2-\lambda) - 3\cdot 2 = -4 + 3\lambda + \lambda^2 = (\lambda+4)(\lambda-1) < \begin{matrix} \lambda_1 = -4 \\ \lambda_2 = 1 \end{matrix}$$

pro $\lambda_1 = -4$ hledáme vl. vektor: $(A - \lambda E)\cdot\vec{u} = \vec{o}$

$$(A + 4E)\cdot\vec{u} = \vec{o}$$

$$\begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \cancel{3}x + \cancel{3}y = 0 \qquad \begin{matrix} y = t \ (t \in \mathbb{R}) \\ x = -t \end{matrix} \rightarrow \vec{u} = \begin{bmatrix} -t \\ t \end{bmatrix} \xrightarrow{t=1} \vec{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\boxed{A\cdot\vec{u} = \lambda\cdot\vec{u}} \qquad \underset{2\times 2}{\begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix}} \cdot \underset{2\times 1}{\begin{bmatrix} -1 \\ 1 \end{bmatrix}} = \begin{bmatrix} 4 \\ -4 \end{bmatrix} = -4\begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \vec{y}_1 = e^{-4x}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -e^{-4x} \\ e^{-4x} \end{bmatrix}$$

vl. vektor pro $\lambda_2 = 1$: $(A-\lambda E)\cdot\vec{v} = \vec{o}$

$$(A-E)\cdot\vec{v} = \vec{o}$$

$$\begin{bmatrix}-2 & 3\\ 2 & -3\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix} \qquad 2x = 3y \qquad y = s \ (s\in\mathbb{R}) \qquad \vec{v} = \begin{bmatrix}\frac{3}{2}s\\ s\end{bmatrix} \xrightarrow{s=2} \vec{v} = \begin{bmatrix}3\\ 2\end{bmatrix}$$

$$2x-3y=0 \qquad x = \frac{3}{2}s$$

zk. $\begin{bmatrix}-1 & 3\\ 2 & -2\end{bmatrix}\begin{bmatrix}3\\ 2\end{bmatrix} = \begin{bmatrix}3\\ 2\end{bmatrix}$

$$\vec{y_2} = e^{x}\begin{bmatrix}3\\ 2\end{bmatrix} = \begin{bmatrix}3e^{x}\\ 2e^{x}\end{bmatrix}$$

OBECNÉ ŘEŠENÍ: $\underbrace{\vec{y} = c_1 e^{-4x}\begin{bmatrix}-1\\ 1\end{bmatrix} + c_2 e^{x}\begin{bmatrix}3\\ 2\end{bmatrix}}_{c_1, c_2 \in \mathbb{R}} = \begin{bmatrix}-c_1 e^{-4x} + 3c_2 e^{x}\\ c_1 e^{-4x} + 2c_2 e^{x}\end{bmatrix}$

$$\vec{y}\,' = \overset{B}{\begin{bmatrix} 13 & -3 & -6 \\ 18 & -2 & -12 \\ 9 & -3 & -2 \end{bmatrix}} \vec{y} \qquad \lambda_1 = 1$$

ch.r. $\quad 0 = \det(B - \lambda E) = \det \begin{bmatrix} 13-\lambda & -3 & -6 \\ 18 & -2-\lambda & -12 \\ 9 & -3 & -2-\lambda \end{bmatrix} =$

$$\begin{matrix} 13-\lambda & -3 & -6 \\ 18 & -2-\lambda & -12 \end{matrix}$$

$$= (13-\lambda)(-2-\lambda)(-2-\lambda) + \underline{324} + \underline{324} - \left(\underline{-54}(-2-\lambda) + \underline{36}(13-\lambda) - \underline{54}(-2-\lambda)\right) =$$

$$= (13-\lambda)(\lambda^2+4\lambda+4) - 36 - 72\lambda = -\lambda^3 + 9\lambda^2 - 24\lambda + 16 = 0$$

$$(\lambda - 1) \cdot \text{NĚCO} = 0$$

$$(-\lambda^3+9\lambda^2-24\lambda+16):(\lambda-1)=-\lambda^2+8\lambda-16$$

$$
\begin{array}{l}
-(-\lambda^3+\lambda^2) \\
\hline
\quad 8\lambda^2-24\lambda+16 \\
\quad -(8\lambda^2-8\lambda) \\
\hline
\quad\quad -16\lambda+16 \\
\quad\quad -(-16\lambda+16) \\
\hline
\quad\quad\quad 0
\end{array}
$$

$$(\lambda-1)\cdot(-\lambda^2+8\lambda-16)=0$$

$\lambda_1=1$ $\quad$ $(4-\lambda)^2=0$

$\lambda_{2,3}=4$

Vl. vektory pro $\lambda_1=1$:

$$(B-\lambda E)\cdot\vec{u}=\vec{o}$$

$$\begin{bmatrix}12 & -3 & -6\\ 18 & -3 & -12\\ 9 & -3 & -3\end{bmatrix}\sim\begin{bmatrix}4 & -1 & -2\\ 6 & -1 & -4\\ 3 & -1 & -1\end{bmatrix}\begin{matrix}\cdot(-3)\\ 2\\ 4\end{matrix}\sim\begin{bmatrix}4 & -1 & -2\\ 0 & 1 & -2\\ \cancel{0} & \cancel{-1} & \cancel{2}\end{bmatrix}$$

$4x-y-2z=0$

$y-2z=0$

$z=t \ (t\in\mathbb{R})$

$y=2t$

$4x=2t+2t\rightarrow x=t$

$$\vec{u}=\begin{bmatrix}t\\ 2t\\ t\end{bmatrix}\xrightarrow{t=1}\vec{u}_1=\begin{bmatrix}1\\ 2\\ 1\end{bmatrix}\rightarrow\vec{y}_1=e^x\begin{bmatrix}1\\ 2\\ 1\end{bmatrix}$$

Vl. vektory pro $\lambda_{2,3} = 4$

$$(B - \lambda E) \cdot \vec{v} = \vec{o}$$

$$\begin{bmatrix} 9 & -3 & -6 \\ \cancel{18} & \cancel{-6} & \cancel{-12} \\ \cancel{9} & \cancel{-3} & \cancel{-6} \end{bmatrix} \cdot \frac{1}{3} \rightarrow 3x - y - 2z = 0$$

$z = t \; (t \in \mathbb{R})$

$x = s \; (s \in \mathbb{R})$

$y = 3s - 2t$

$$\vec{v} = \begin{bmatrix} s \\ 3s - 2t \\ t \end{bmatrix} = \begin{bmatrix} s \\ 3s \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -2t \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

$$\vec{y}_2 = e^{4x} \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} \qquad \vec{y}_3 = e^{4x} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

$$\vec{y} = c_1 e^{x} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + c_2 e^{4x} \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} + c_3 e^{4x} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \quad c_1, c_2, c_3 \in \mathbb{R}$$

$$\vec{y}\,' = \overset{C}{\begin{bmatrix} -13 & 4 & 8 \\ -24 & 7 & 14 \\ -13 & 4 & 8 \end{bmatrix}} \cdot \vec{y}$$

$$0 = \det(C - \lambda E) = \det \begin{bmatrix} -13-\lambda & 4 & 8 \\ -24 & 7-\lambda & 14 \\ -13 & 4 & 8-\lambda \end{bmatrix} =$$

$$\begin{matrix} -13-\lambda & 4 & 8 \\ -24 & 7-\lambda & 14 \end{matrix}$$

$$= (-13-\lambda)(7-\lambda)(8-\lambda) - 768 - 728 - \left(-104(7-\lambda) + 56(-13-\lambda) - \overset{96}{\cancel{48}}(8-\lambda)\right) =$$

$$= -\lambda^3 + 2\lambda^2 - 5\lambda = 0 = -\lambda(\lambda^2 - 2\lambda + 5) \qquad \boxed{\lambda_1 = 0}$$

$$\lambda^2 - 2\lambda + 5 = 0$$

$$\lambda_{2,3} = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = \underline{\underline{1 \pm 2i}}$$

pro $\lambda_1 = 0$ je vl. vektor $\vec{u} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \rightarrow \vec{y_1} = e^{0x} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$

$(\vec{v} = (2; 3+i, 2))$

pro $\lambda_2 = 1+2i$: $(C - \lambda E) \cdot \vec{v} = \vec{o}$

$$\begin{array}{c} 13 \\ \\ (-14-2i) \end{array} \begin{bmatrix} -14-2i & 4 & 8 \\ -24 & 6-2i & 14 \\ -13 & 4 & 7-2i \end{bmatrix} \begin{array}{c} 24 \\ (-14-2i) \\ \\ \end{array} \sim \begin{bmatrix} -14-2i & 4 & 8 \\ 0 & 8+16i & -4-28i \\ 0 & -14-8i & 2+14i \end{bmatrix} \begin{array}{c} \\ (14+8i) \\ (8+16i) \end{array}$$

$$96 + (6-2i)(-14-2i) = 96 - 84 - 12i + 28i - 4 = 8 + 16i$$

$$104 + (7-2i)(-14-2i) = 104 - 98 - 14i + 28i - 4 = 2$$

?

$$(-4-28i)(14+8i) + (8+16i)(2+14i) = -56 - \cancel{32i} - 392i + \cancel{224} + 16 + 112i + \cancel{32i} - \cancel{224} =$$
$$= -40$$