$$\vec{y}\,' = \begin{bmatrix} 6 & -6 \\ 2 & -1 \end{bmatrix}\vec{y} + \begin{bmatrix} 6 \\ 2 \end{bmatrix} e^{2x}, \quad \vec{y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\vec{y}_H = e^{2t}\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot c_1 + c_2 e^{3t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$c_1, c_2 \in \mathbb{R}$

a) $\vec{y}\,' = \overset{A}{\begin{bmatrix} 6 & -6 \\ 2 & -1 \end{bmatrix}}\vec{y}$

$$0 = \det(A - \lambda E) = \det\begin{bmatrix} 6-\lambda & -6 \\ 2 & -1-\lambda \end{bmatrix} = (6-\lambda)(-1-\lambda) + 12 =$$

$$= \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0 \quad < \begin{matrix} \lambda_1 = 2 \\ \lambda_2 = 3 \end{matrix}$$

$\lambda_1 = 2$: $\begin{bmatrix} 4 & -6 \\ 2 & -3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \vec{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$2x - 3y = 0 \quad y = 2t \ (t \in \mathbb{R})$

$x = 3t$

$\lambda_2 = 3$

$\begin{bmatrix} 3 & -6 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad x - 2y = 0 \quad y = s \ (s \in \mathbb{R})$

$x = 2y \quad x = 2s$

b) $\vec{y}\,' = \begin{bmatrix} 6 & -6 \\ 2 & -1 \end{bmatrix} \vec{y} + \begin{bmatrix} 6 \\ 2 \end{bmatrix} e^{2x}$

$\alpha = 2 \rightarrow k = 1$

(2 je jednonásobný kořen ch.r.)

$\vec{y}_p = e^{2x} \begin{bmatrix} Ax + B \\ Cx + D \end{bmatrix}$ $\quad \vec{y}_p\,' = e^{2x} \cdot 2 \begin{bmatrix} Ax + B \\ Cx + D \end{bmatrix} + e^{2x} \begin{bmatrix} A \\ C \end{bmatrix}$

dosazení: $2\cancel{e^{2x}} \begin{bmatrix} Ax + B \\ Cx + D \end{bmatrix} + \cancel{e^{2x}} \begin{bmatrix} A \\ C \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} Ax + B \\ Cx + D \end{bmatrix} \cancel{e^{2x}} + \begin{bmatrix} 6 \\ 2 \end{bmatrix} \cancel{e^{2x}}$

1. řádek: $2Ax + 2B + A = 6(Ax + B) - 6(Cx + D) + 6$

2. řádek: $2Cx + 2D + C = 2(Ax + B) - (Cx + D) + 2$

1. řádek u x: $2A = 6A - 6C$

u x^0: $A + 2B = 6B - 6D + 6$

2. řádek: u x: $2C = 2A - C$

u x^0: $2D + C = 2B - D + 2$

$4A - 6C = 0$ ✓

$A - 4B + 6D = 6$ ✓

$2A - 3C = 0$ ✓

$2B - C - 3D = -2$ ✓

$$\left[\begin{array}{cccc|c} 2 & 0 & -3 & 0 & 0 \\ 1 & -4 & 0 & 6 & 6 \\ 0 & 2 & -1 & -3 & -2 \end{array}\right] \cdot(-2) \sim \left[\begin{array}{cccc|c} 2 & 0 & -3 & 0 & 0 \\ 0 & 8 & -3 & -12 & -12 \\ 0 & \underline{2} & -1 & -3 & -2 \end{array}\right] \cdot(-4) \sim$$

$$\sim \left[\begin{array}{cccc|c} 2 & 0 & -3 & 0 & 0 \\ 0 & 8 & -3 & -12 & -12 \\ 0 & 0 & 1 & 0 & -4 \end{array}\right]$$

$2A - 3C = 0$

$8B - 3C - 12D = -12$

$C = -4$

$\underline{C = -4}$ $\underline{A = -6}$ $\underline{D = t \; (t \in \mathbb{R})}$

$8B + 12 - 12t = -12$

$8B = -24 + 12t$

$\underline{B = -3 + \frac{3}{2}t}$

$$\vec{y}_p = e^{2x}\begin{bmatrix} Ax + B \\ Cx + D \end{bmatrix} = e^{2x}\begin{bmatrix} -6x - 3 + \frac{3}{2}t \\ -4x + t \end{bmatrix} =$$

$$= e^{2x}\begin{bmatrix} -6x - 3 \\ -4x \end{bmatrix} + e^{2x}\begin{bmatrix} +\frac{3}{2} \\ 1 \end{bmatrix} \cdot t \overset{t=0}{=\!=} e^{2x}\begin{bmatrix} -6x - 3 \\ -4x \end{bmatrix}$$

~~x má být t~~
~~parametr je potřeba~~
~~zapsat jinak~~

obecné řešení: $\vec{y} = \vec{y}_H + \vec{y}_p = c_1 e^{2t}\begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^{2t}\begin{bmatrix} -6t-3 \\ -4t \end{bmatrix}$

$\vec{y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$: $\quad \begin{bmatrix} 1 \\ 2 \end{bmatrix} = c_1\begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \end{bmatrix}$

$$3c_1 + 2c_2 = 4$$
$$2c_1 + c_2 = 2 \quad /(-2)$$

$$-c_1 = 0 \rightarrow \boxed{\begin{matrix} c_1 = 0 \\ c_2 = 2 \end{matrix}}$$

$$\boxed{\vec{y} = 2e^{3t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^{2t}\begin{bmatrix} -6t-3 \\ -4t \end{bmatrix}}$$

$$\vec{y}\,' = \begin{bmatrix}-2 & 2\\ -1 & 1\end{bmatrix}\cdot\vec{y} + \begin{bmatrix}x\\ 2\end{bmatrix} e^{0x}$$

a) homogenní: $\lambda_1 = -1$ $\quad \vec{v}_1 = \begin{bmatrix}2\\ 1\end{bmatrix}$

$\lambda_2 = 0$ $\quad \vec{v}_2 = \begin{bmatrix}1\\ 1\end{bmatrix}$

$$\vec{y}_H = c_1\cdot e^{-x}\begin{bmatrix}2\\ 1\end{bmatrix} + c_2\begin{bmatrix}1\\ 1\end{bmatrix}$$

$\alpha = 0 \to k = 1$

$$\vec{y}_p = \begin{bmatrix}Ax^2+Bx+C\\ Dx^2+Ex+F\end{bmatrix} \qquad \vec{y}_p\,' = \begin{bmatrix}2Ax+B\\ 2Dx+E\end{bmatrix}$$

dosazení: $$\begin{bmatrix}2Ax+B\\ 2Dx+E\end{bmatrix} = \begin{bmatrix}-2 & 2\\ -1 & 1\end{bmatrix}\begin{bmatrix}Ax^2+Bx+C\\ Dx^2+Ex+F\end{bmatrix} + \begin{bmatrix}x\\ 2\end{bmatrix}$$

1. řádek: $2Ax + B = -2Ax^2 - 2Bx - 2C + 2Dx^2 + 2Ex + 2F + x$

2. řádek: $2Dx + E = -Ax^2 - Bx - C + Dx^2 + Ex + F + 2$

1\. řádek $ux^2: \cancel{0 = -2A + 2D}$

$ux: 2A = -2B + 2E + 1$

$ux^0: B = -2C + 2F$

2\. řádek $ux^2: 0 = -A + D$ ✓

$ux: 2D = -B + E$

$ux^0: E = -C + F + 2$

$$-A + D = 0 \rightarrow \boxed{A = D}$$

$$2A + 2B - 2E = 1 \quad *$$

$$B + 2C - 2F = 0$$

$$B + 2D - E = 0 \quad **$$

$$C + E - F = 2$$

$$2B + 2D - 2E = 1 \quad *$$

$$B + 2D - E = 0 \quad ** \; (-2)$$

$$-2D = 1 \rightarrow D = A = -\frac{1}{2}$$

$$\left.\begin{aligned} 2B - 1 - 2E &= 1 \\ B - 1 - E &= 0 \end{aligned}\right\} \boxed{B - E = 1}$$

$$B - E = 1$$

$$B + 2C - 2F = 0$$

$$C + E - F = 2$$

$\cancel{\boxed{E = 0}}$ (zvolíme)

$\cancel{\boxed{B = 1}}$

$\cancel{2C - 2F = -1}$

$\cancel{C - F = 2}$

$$B - E = 1$$
$$B + 2C - 2F = 0$$
$$C + E - F = 2$$

$$\begin{array}{cccc|c} B & C & E & F & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 1 \\ 1 & 2 & 0 & -2 & 0 \\ 0 & 1 & 1 & -1 & 2 \end{array}\right] \cdot(-1) \sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 1 \\ 0 & 2 & 1 & -2 & -1 \\ 0 & 1 & 1 & -1 & 2 \end{array}\right] \cdot(-2) \sim$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 1 \\ 0 & 2 & 1 & -2 & -1 \\ 0 & 0 & -1 & 0 & -5 \end{array}\right]$$

$$B - E = 1 \qquad \boxed{B = 6}$$
$$2C + E - 2F = -1 \qquad \boxed{F = 0} \text{ zvolíme}$$
$$-E = -5 \rightarrow \boxed{E = 5}$$

$$2C + 5 = -1$$
$$2C = -6 \qquad \boxed{A = -\tfrac{1}{2}}$$
$$\boxed{C = -3} \qquad \boxed{D = -\tfrac{1}{2}}$$

$$\vec{y}_p = \begin{bmatrix} Ax^2 + Bx + C \\ Dx^2 + Ex + F \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x^2 + 6x - 3 \\ -\frac{1}{2}x^2 + 5x \end{bmatrix}$$

$$\vec{y} = \vec{y}_H + \vec{y}_p = c_1 e^{-x} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2}x^2 + 6x - 3 \\ -\frac{1}{2}x^2 + 5x \end{bmatrix}$$

$$\vec{y}\,' = \overset{C}{\begin{bmatrix} 1 & 2 & -1 \\ -1 & 2 & 1 \\ -1 & -1 & 4 \end{bmatrix}} \vec{y} + \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} e^{-x}$$

$\lambda_1 = 3 \quad \vec{v}_1 = (1,3,4)$

$\lambda_{2,3} = 2 \quad \vec{v}_2 = (1,1,1)$

$\vec{y}_1 = e^{3x}\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \quad \vec{y}_2 = e^{2x}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

zobecnený vl. vektor pre $\lambda_2 = 2$

$$(C-2E)^2 \cdot \vec{u} = \vec{o}$$

$$\begin{bmatrix} -1 & 2 & -1 \\ -1 & 0 & 1 \\ -1 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} -1 & 2 & -1 \\ -1 & 0 & 1 \\ -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 0 & -3 & 3 \\ 0 & -4 & 4 \end{bmatrix}$$

$-y + z = 0$

$y = z = t$

$x = s$

$$\vec{u} = \begin{bmatrix} s \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \left(\text{zvolene } \begin{matrix} s=1 \\ t=0 \end{matrix}\right)$$

$$\vec{y}_3 = e^{2x}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + x\begin{bmatrix} -1 & 2 & -1 \\ -1 & 0 & 1 \\ -1 & -1 & 2 \end{bmatrix}\right)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = e^{2x}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}\right) = e^{2x}\begin{bmatrix} 1-x \\ -x \\ -x \end{bmatrix}$$

$$\vec{y}\,' = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 2 & 1 \\ -1 & -1 & 4 \end{bmatrix} \vec{y} + \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} e^{-x}$$

$$\vec{y}_p = e^{-x} \begin{bmatrix} A \\ B \\ C \end{bmatrix} \qquad \vec{y}_p\,' = -e^{-x} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

$$\alpha = -1 \longrightarrow k = 0$$

dosazení: $-\cancel{e^{-x}} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 2 & 1 \\ -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} \cancel{e^{-x}} + \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \cancel{e^{-x}}$

$$\begin{aligned} -A &= A + 2B - C + 2 \\ -B &= -A + 2B + C + 2 \\ -C &= -A - B + 4C + 1 \end{aligned} \longrightarrow \boxed{\begin{aligned} 2A + 2B - C &= -2 \\ -A + 3B + C &= -2 \\ -A - B + 5C &= -1 \end{aligned}}$$

$$\left[\begin{array}{ccc|c} 2 & 2 & -1 & -2 \\ -1 & 3 & 1 & -2 \\ -1 & -1 & 5 & -1 \end{array}\right] \begin{array}{c} \\ (2) \\ (2) \end{array} \sim \left[\begin{array}{ccc|c} 2 & 2 & -1 & -2 \\ 0 & 8 & 1 & -6 \\ 0 & 0 & 9 & -4 \end{array}\right]$$

$$2A + 2B - C = -2$$

$$8B + C = -6 \to 8B = -6 - \frac{4}{9} = -\frac{58}{9} \to B = -\frac{58}{72}$$

$$9C = -4 \to C = -\frac{4}{9}$$

$$2A = -2 + \frac{58}{36} - \frac{4}{9} = \frac{-72+58-16}{36} = \frac{-30}{36}$$

$$A = \frac{-15}{36} = \frac{-5}{12}$$

$$\vec{y_p} = \begin{bmatrix} A \\ B \\ C \end{bmatrix} e^{-x} = \begin{bmatrix} -\frac{5}{12} \\ -\frac{58}{72} \\ -\frac{4}{9} \end{bmatrix} e^{-x}$$

$$\vec{y} = \vec{y}_H + \vec{y}_p = c_1 e^{3x} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + c_2 e^{2x} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{2x} \begin{bmatrix} 1-x \\ -x \\ -x \end{bmatrix} + e^{-x} \begin{bmatrix} -\frac{5}{12} \\ -\frac{58}{72} \\ -\frac{4}{9} \end{bmatrix}$$